Cuntz' proof of Bott periodicity for C^*-algebras

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Overview. Today we will talk about Cuntz' simple proof of Bott periodicity of C^*-algebra K-theory [Cun84]. It has the following advantages:

- It is simpler and shorter than any proof of Bott periodicity for topological K-theory.
- The definitions of K_0 and K_1 themselves are not directly used. The proof only relies on properties of these functors (homotopy invariance, half-exactness, stability with respect to $\cdot \otimes \mathcal{K}$). This makes the proof very general.

Notation. Let $\mathtt{C}^*$ be the category of C^*-algebras with $*$-homomorphisms (recall that these are automatically continuous). Let also $\mathtt{Ab}$ be any abelian category, e.g. the category of abelian groups with group homomorphisms.

For $A \in \mathtt{C}^*$, We consider the cylinder over A,

$$MA \equiv \left\{ f : [0,1] \stackrel{\text{cont.}}{\longrightarrow} A \right\}$$

and the suspension of A,

$$SA \equiv \left\{ f : [0,1] \stackrel{\text{cont.}}{\longrightarrow} A \;\middle|\; f(0) = f(1) = 0 \right\}.$$

Both M and S are functors $\mathtt{C}^* \to \mathtt{C}^*$.

As usual, $\mathcal{K}$ denotes the C^*-algebra of compact operators on $\ell^2(\mathbb{N})$, and $e \in \mathcal{K}$ is any fixed projection of rank 1.

Half-exactness and long exact sequences. the category of abelian groups with group homomorphisms. We will consider the following properties of functors $E : \mathtt{C}^* \to \mathtt{Ab}$:

(a) E is half-exact, i.e. it maps any extension of C^*-algebras

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0 \tag{1}$$

to an exact sequence

$$E(J) \longrightarrow E(A) \longrightarrow E(A/J) \; .$$

(In general, this need not be a short exact sequence.)

(b) E is homotopy invariant: for any $A \in \mathtt{C}^*$, the functor E assigns to the evaluation map

$$\mathrm{ev}_0 : MA \longrightarrow A, \qquad\qquad a \longmapsto a(0),$$

a morphism $E(\mathrm{ev}_0)$ which is an isomorphism. (This is equivalent to the usual definition of homotopy invariance.)

(c) E is $\mathcal{K}$-stable, i.e. if for any $A \in \mathtt{C}^*$, the inclusion $*$-homomorphism

$$\kappa : A \longrightarrow A \otimes \mathcal{K}, \qquad\qquad a \longmapsto a \otimes e$$

yields an isomorphism $E(\kappa)$.

We have seen previously that both K_0 and K_1 have all these properties. It will turn out that rather than using the definition of the K-theory invariants K_0 and K_1 themselves, properties (a)-(c) alone are actually enough for proving Bott periodicity. In bivariant K-theory, there are plenty of other functors having these properties (e.g. $KK(B, \cdot)$ for $B \in \mathtt{C}^*$ satisfying certain nuclearity conditions [Kas81, §7 thm. 2], [Ska88, 3.6], or $E(B, \cdot)$ for any $B \in \mathtt{C}^*$ [Bla86, 25]). Hence the present proof of Bott periodicity also applies to these other functors.

Theorem 1 (e.g. [Kas81, §7 l. 5])**.** *Let E satisfy (a) and (b). Then E has long exact sequences: for any extension (1), there is a long exact sequence*

$$\ldots \longrightarrow E(S^2(A/J)) \xrightarrow{\delta_2} E(SJ) \longrightarrow E(SA) \longrightarrow E(S(A/J)) \xrightarrow{\delta_1} E(J) \longrightarrow E(A) \longrightarrow E(A/J)$$

with boundary maps δ_n which are natural in the extension.

We are going to need only the last few terms of this sequence. The exactness of these has been established in the preceding lectures for the functor $E = K_0$.

Corollary 2. *If the extension (1) is split, any splitting induces a direct sum decomposition*

$$E(A) \cong E(J) \oplus E(A/J).$$

The Toeplitz algebra. The main idea of the proof is to use the long exact sequence on an appropriate extension of C^*-algebras. We will soon be able to define this extension and use it for showing that $E(S^2A) \cong E(A)$ first for $A = \mathbb{C}$ and then generally. This extension will involve the following C^*-algebra:

Definition 3 (Toeplitz algebra)**.** *Let $\mathcal{H} = \ell^2(\mathbb{N})$ be the Hilbert space of square-summable sequences, and $\xi_n \in \mathcal{H}$ the nth standard basis vector. The isometry*

$$v : \mathcal{H} \to \mathcal{H}, \qquad \xi_n \mapsto \xi_{n+1},$$

together with its hermitian conjugate

$$v^* : \mathcal{H} \to \mathcal{H}, \qquad \xi_n \mapsto \begin{cases} \xi_{n-1} & \text{for } n > 1, \\ 0 & \text{for } n = 1. \end{cases}$$

and all polynomials in v and v^* generate a C^*-algebra $C^*(v)$, the ***Toeplitz algebra*** *$\mathcal{T} \equiv C^*(v)$.*

Theorem 4 (Coburn)**.** *The Toeplitz algebra $\mathcal{T} = C^*(v)$ is the universal unital C^*-algebra generated by an isometry: for any unital $A \in \mathtt{C}^*$ and any isometry*[1] *$w \in A$, there is a unique unital $*$-homomorphism*

$$C^*(v) \xrightarrow{v \mapsto w} A$$

Proof. This is basically contained in [Cob67]. □

In the following, homomorphisms from $\mathcal{T}$ to any other C^*-algebra will always be defined via this universal property. In fact, one could also regard the universal property of $\mathcal{T}$ as its **definition**, so that definition 3 then becomes the theorem. One can also use the universal property to define non-unital $*$-homomorphisms from $\mathcal{T}$ to any other C^*-algebra: let $p \in A$ be any projection and $w \in A$ any element with $w^*w = p$ and $ww^* \leq p$. Then the C^*-subalgebra $pAp \subseteq A$ is unital with p as the unit and contains w as an isometry, and hence we obtain an induced $*$-homomorphism $\mathcal{T} \longrightarrow A$ which maps 1 to p and v to w.

The Toeplitz extension. $\mathcal{C}(S^1)$ denotes the C^*-algebra of continuous functions on the unit circle.

Proposition 5. *There is an extension*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow \mathcal{C}(S^1) \longrightarrow 0$$

Proof. Note that $1 - vv^*$ is the projection onto the first standard basis vector ξ_1. In a similar vein, $1 - v^n v^{*n}$ is the projection onto the subspace $\mathrm{lin}(\xi_1, \ldots, \xi_n) \subseteq \mathcal{H}$. Hence $\mathcal{T}$ contains any projection onto any ξ_n. By applying v or v^* an appropriate number of times, one can obtain any rank 1 operator of the form

$$\eta \longmapsto \langle \xi_n, \eta \rangle \, \xi_m. \tag{2}$$

But since these operators generate $\mathcal{K}$, we have that $\mathcal{K} \cap \mathcal{T} = \mathcal{K}$. On the other hand, $\mathcal{K}$ is an ideal in $\mathcal{B}(\mathcal{H})$, and therefore also an ideal in $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H})$.

In order to determine the quotient $\mathcal{T}/\mathcal{K}$, note that $1 - vv^*$ lies in $\mathcal{K}$. With the quotient map

$$\pi : \mathcal{T} \twoheadrightarrow \mathcal{T}/\mathcal{K}$$

the image $\pi(v)$ is therefore a unitary operator. Since v generates $\mathcal{T}$, it is clear that $\pi(v)$ generates $\mathcal{T}/\mathcal{K}$. Therefore, $\mathcal{T}/\mathcal{K}$ is isomorphic to the C^*-algebra of continuous functions on $\mathrm{spec}(\pi(v))$.

Actually, since $\mathcal{T}/\mathcal{K}$ is the abelianization of $\mathcal{T}$, it is a straightforward consequence of theorem 4 that $\mathcal{T}/\mathcal{K}$ is the universal unital C^*-algebra generated by the unitary element $\pi(v)$. In particular, for every $\lambda \in S^1 \subseteq \mathbb{C}$ there is a unital $*$-homomorphism from $\mathcal{T}/\mathcal{K}$ to $\mathbb{C}$ which maps $\pi(v)$ to λ. Hence $\mathrm{spec}(\pi(v)) = S^1$. □

[1]recall that an operator w is defined to be an isometry if and only if $w^*w = 1$, which is equivalent to the requirement that w preserves the norm of all vectors in the Hilbert space it acts on.

Tensor products of C^*-algebras.

Definition 6. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ and $B \subseteq \mathcal{B}(\mathcal{H})$ be concretely represented C^*-algebras. The two inclusions*

$$A \hookrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H}), \qquad a \longmapsto a \otimes 1$$

$$B \hookrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H}), \qquad b \longmapsto 1 \otimes b$$

generate a C^*-algebra denoted by $A \otimes B$.

Alternatively, one can say that $A \otimes B$ is a suitable completion of the algebraic tensor product $A \otimes_{\text{alg}} B$, which is taken in the category of $\mathbb{C}$-algebras. It is a basic result [KR97, 11.1] that the isomorphism type of $A \otimes B$ does not depend on the concrete embeddings in $\mathcal{B}(\mathcal{H})$. As an example of a tensor product, one has that for any C^*-algebra A,

$$A \otimes S\mathbb{C} \cong SA.$$

Finally, we will use the fact that a simple application of nuclearity theory [Bla86, 15.8] shows that one can take a tensor product of the Toeplitz extension by any other $A \in \mathtt{C}^*$ to obtain a new extension

$$0 \longrightarrow \mathcal{K} \otimes A \longrightarrow \mathcal{T} \otimes A \longrightarrow \mathcal{C}(S^1) \otimes A. \longrightarrow 0 \tag{3}$$

Calculating $E(\mathcal{T})$. The main burden of the Bott periodicity proof consists in calculating $E(\mathcal{T})$.

Lemma 7. *Let $f, g : A \longrightarrow B$ be $*$-homomorphisms such that $f + g$ is also a $*$-homomorphism. Then for any E as above,*

$$E(f + g) = E(f) + E(g). \tag{4}$$

Proof. Note that $f + g$ is a $*$-homomorphism if and only if the ranges of f and g are orthogonal, i.e. $\operatorname{im}(f) \cdot \operatorname{im}(g) = 0$. Then the C^*-subalgebras $\operatorname{im}(f) \subseteq B$ and $\operatorname{im}(g) \subseteq B$ also embed as a direct sum,

$$\operatorname{im}(f) \oplus \operatorname{im}(g) \subseteq B.$$

With the identification $E(\operatorname{im}(f) \oplus \operatorname{im}(g)) \cong E(\operatorname{im}(f)) \oplus E(\operatorname{im}(g))$,

$$\begin{tikzcd}
 & E(\operatorname{im}(f)) \\
E(A) \arrow[ur, "E(f)"] \arrow[r, "\beta"] \arrow[dr, "E(g)"'] & E(\operatorname{im}(f)) \oplus E(\operatorname{im}(g)) \arrow[u] \arrow[d] \\
 & E(\operatorname{im}(g))
\end{tikzcd}$$

there is exactly one homomorphism β making this diagram commute. Now both sides of (4) have this property. $\square$

Now consider the $*$-homomorphism

$$j : \mathbb{C} \longrightarrow \mathcal{T}, \qquad 1 \longmapsto 1\ ,$$

which simply is the inclusion of the identity, and the $*$-homomorphism

$$q : \mathcal{T} \longrightarrow \mathbb{C}, \qquad S \longmapsto 1$$

Proposition 8. *Let E be a functor satisfying all the above properties (a)-(c). Then $E(j)$ and $E(q)$ are mutually inverse isomorphisms, so that $E(\mathcal{T}) \cong E(\mathbb{C})$.*

Proof. By construction, $q \circ j = \mathrm{id}_{\mathbb{C}}$, hence

$$E(q) \circ E(j) = \mathrm{id}_{E(\mathbb{C})}.$$

Therefore, $E(\mathbb{C})$ is a direct summand of $E(\mathcal{T})$.

The converse direction is more difficult. We work with the (partially commutative) diagram

$$\begin{tikzcd}
 & \mathcal{T} \arrow[d, hook, "\kappa:"', "v\mapsto e\otimes v"] & & & \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{T} \arrow[r, "\iota"'] \arrow[d, equal] & \overline{\mathcal{T}} \arrow[r, "p"'] \arrow[d, "r"] & \mathcal{T} \arrow[r] \arrow[d, "v\mapsto\pi(v)\otimes 1"] \arrow[l, bend right, "s"'] \arrow[ull, bend right, "j\circ q"'] \arrow[ull, bend right=20, equal] & 0 \\
0 \arrow[r] & \mathcal{K}\otimes\mathcal{T} \arrow[r] & \mathcal{T}\otimes\mathcal{T} \arrow[r] & \mathcal{C}(S^1)\otimes\mathcal{T} \arrow[r] & 0
\end{tikzcd}$$

which will now be explained step by step. The bottow row is a short exact sequence according to (3). The C^*-algebra $\overline{\mathcal{T}}$ is defined such that the right square is a pullback diagram, or, equivalently, such that the middle row is also a short exact sequence. We now get to the definition of the section s. Firstly, consider the $*$-homomorphism

$$\alpha : \mathcal{T} \longrightarrow \mathcal{T}\otimes\mathcal{T}, \qquad v \longmapsto v(1-e)\otimes 1\ .$$

Since the image of $\alpha(v)$ in quotient $\mathcal{C}(S^1)\otimes\mathcal{T}$ is also $\pi(v)\otimes 1$, we obtain s via the universal property of the pullback $\overline{\mathcal{T}}$. This s satisfies $p\circ s$ by construction. Hence we obtain a splitting

$$E(\overline{\mathcal{T}}) = E(\mathcal{K}\otimes\mathcal{T}) \oplus E(\mathcal{T})$$

which shows in particular that $E(\iota)$ is injective. We register this partial result for later use.

Our main interest lies in the two uppermost arrows, which are $j\circ q$ and $\mathrm{id}_{\mathcal{T}}$. Our goal is to show that they coincide after application of E. To this end, consider the morphisms

$$\begin{aligned}
\phi_0 &\equiv \alpha + r\,\iota\,\kappa \circ \mathrm{id}_{\mathcal{T}}, \qquad & \phi_0(v) &= v(1-e)\otimes 1 + e\otimes v,\\
\phi_1 &\equiv \alpha + r\,\iota\,\kappa \circ (j\circ q), \qquad & \phi_1(v) &= v(1-e)\otimes 1 + e\otimes 1.
\end{aligned}$$

The operators $\phi_i(v)$ are isometries on $\mathcal{H} \otimes \mathcal{H}$ having the form $\phi_i(v) = u_i(v \otimes 1)$ for the self-adjoint unitaries

$$
\begin{aligned}
u_0 \equiv & v(1-e)v^* \otimes 1 + ev^* \otimes v + ve \otimes v^* + e \otimes e \\
u_1 \equiv & v(1-e)v^* \otimes 1 + ev^* \otimes 1 + ve \otimes 1.
\end{aligned}
$$

(See figure 1 for an illustration of $\phi(v_i)$ and u_i.) By continuously moving the -1 eigenvalue of u_0 to a $+1$ eigenvalue via functional calculus along the complex unit circle, we can find a continuous path of unitaries connecting u_0 to the identity operator. Since the same applies to u_1, the operators u_0 and u_1 are actually homotopic as unitaries in the C^*-algebra $\mathcal{T} \otimes \mathcal{T}$ via a path $t \mapsto u_t$ which has the property that all u_t map to $\pi(v) \otimes 1$ in $\mathcal{C}(S^1) \otimes \mathcal{T}$. Hence the ϕ_i can be connected via a continuous path of $*$-homomorphisms ϕ_t which make the subdiagram

$$
\begin{tikzcd}
 & \mathcal{T} \arrow[dl, "\phi_t"'] \arrow[d, "v \mapsto \pi(v) \otimes 1"] & \\
\mathcal{T} \otimes \mathcal{T} \arrow[r] & \mathcal{C}(S^1) \otimes \mathcal{T} \arrow[r] & 0
\end{tikzcd}
$$

commute for all t. But then the universal property of $\overline{\mathcal{T}}$ shows that the $*$-homomorphisms

$$
s + \iota\, \kappa \circ \mathrm{id}_{\mathcal{T}} \qquad \text{and} \qquad s + \iota\, \kappa \circ (j \circ q)
$$

are also homotopic, which implies that

$$
E\left(s + \iota\, \kappa \circ \mathrm{id}_{\mathcal{T}}\right) = E\left(s + \iota\, \kappa \circ (j \circ q)\right)
$$

by homotopy invariance of E.

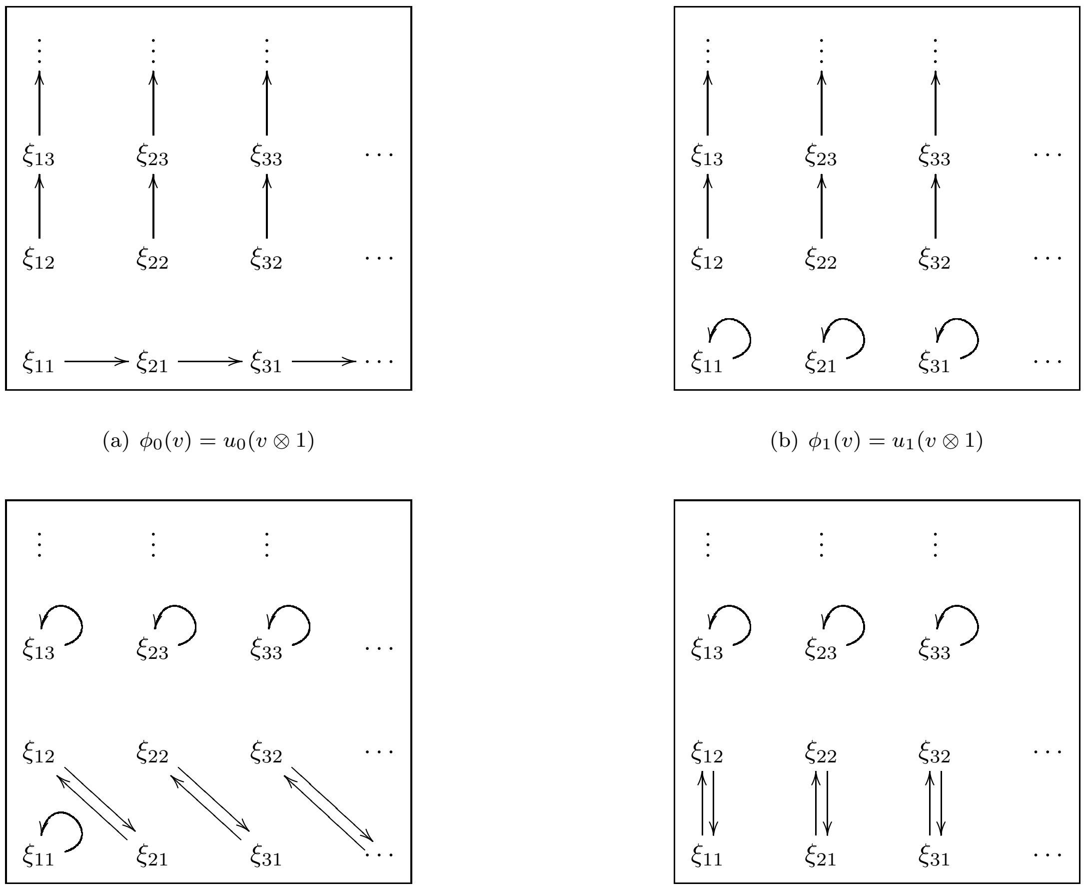
But now due to additivity (4) and functoriality of E, we also have

$$
E(\iota) \circ E(\kappa) \circ \mathrm{id}_{E(\mathcal{T})} = E(\iota) \circ E(\kappa) \circ E(j) \circ E(q).
$$

Now the desired results follows from injectivity of $E(\iota)$ and the assumption that $E(\kappa)$ is an isomorphism. $\square$

Theorem 9 (Bott periodicity)**.** *For any functor E satisfying the above properties (a)-(c), there is a natural isomorphism $E(S^2 A) \cong E(A)$.*

Proof. We start with the case $A = \mathbb{C}$. Note that we can think of $S\mathbb{C}$ as those functions in $\mathcal{C}(S^1)$



(a) $\phi_0(v) = u_0(v \otimes 1)$

(b) $\phi_1(v) = u_1(v \otimes 1)$

(c) u_0

(d) u_1

Figure 1: Illustration of the isometries and unitaries involved in the proof of proposition 8. The vectors $\xi_{nm} \equiv \xi_n \otimes \xi_m$ are the standard basis of $\mathcal{H} \otimes \mathcal{H} = \ell^2(\mathbb{N} \times \mathbb{N})$.

which vanish on some basepoint. Then consider the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, equal] & \mathcal{T}_0 \arrow[r] \arrow[d] & S\mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{T} \arrow[r] \arrow[d, "j"] & \mathcal{C}(S^1) \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{C} \arrow[r, equal] \arrow[d] & \mathbb{C} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{5}
$$

where $\mathcal{T}_0$ is defined as the kernel of j. Then by the nine lemma for C^*-algebras, also the top row is exact. Applying the long exact sequence to the defining extension of $\mathcal{T}_0$ yields that all invariants of $\mathcal{T}_0$ are trivial,

$$E(S^n\mathcal{T}_0) = 0 \quad \forall n \in \mathbb{N}_0.$$

But now an application of the long exact sequence to the two row of (5) yields in particular

$$
\begin{tikzcd}
\cdots \arrow[r] & E(S^{n+1}\mathcal{T}_0) \arrow[r] \arrow[d, equal] & E(S^{n+1}S\mathbb{C}) \arrow[r] \arrow[d, equal] & E(S^n\mathcal{K}) \arrow[r] \arrow[d, "\cong"] & E(S^n\mathcal{T}_0) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & E(S^{n+2}\mathbb{C}) \arrow[r] & E(S^n\mathbb{C}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

where the first row is easily evaluated as being isomorphic to the second row. Hence by exactness, $E(S^{n+2}\mathbb{C}) \cong E(S^n\mathbb{C})$ for all $n \in \mathbb{N}_0$, which is the desired statement.

Now to the case of general A. This can be done with exactly the same steps, beginning with proposition 8, expect that now all the statements and diagrams have to be taken $\cdot \otimes A$. For example, one starts by copying the proof of proposition 8 to show that $E(j \otimes \mathrm{id}_A)$ and $E(q \otimes \mathrm{id}_A)$ are mutually inverse isomorphisms. One then proceeds to show that $E\left(S^n(\mathcal{T}_0 \otimes A)\right) = 0$, and so on.

□

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