# Quantum correlations and group $C^*$-algebras

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**Outline:**

(a) Why quantum correlations are weird: Hardy's nonlocality paradox,

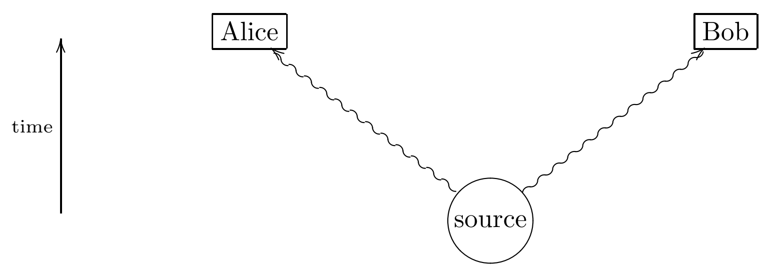
(b) Brief recap of quantum kinematics,

(c) The relevance of group $C^*$-algebras,

(d) Results for two iterated binary measurements,

(e) Outlook: From Kirchberg's conjectures to Tsirelson's problem.

**Hardy's nonlocality paradox as a possibilistic Bell inequality.** The most basic scenario for demonstrating what quantum correlations are is the following: there are two parties, commonly dubbed **Alice** and **Bob**, experimenting in distant labs. They each receive a physical system prepared in some way, e.g. a photon, from a common source:




In their labs, it is understood that they can conduct certain measurements on their respective system, and for each run they are free to choose one of these measurements.

Let us say for the sake of concreteness the available measurements measure the properties "color" and "shape", and these can take on the values:

$$\text{color} \in \{\text{red}, \text{blue}\}$$
$$\text{shape} \in \{\text{round}, \text{square}\}.$$

Assume that when they both measure color, it sometimes happens that they both find red[1]

$$\diamond\,(\text{Alice: color} \to \text{red}, \qquad \text{Bob: color} \to \text{red}) \tag{1}$$

On the other hand, assume that some joint outcomes are impossible in some coincidence measurements:

$$\begin{aligned} &\neg\diamond\,(\text{Alice: color} \to \text{red}, \qquad \text{Bob: shape} \to \text{square}) \\ &\neg\diamond\,(\text{Alice: shape} \to \text{square}, \qquad \text{Bob: color} \to \text{red}) \\ &\neg\diamond\,(\text{Alice: shape} \to \text{round}, \qquad \text{Bob: shape} \to \text{round}) \end{aligned} \tag{2}$$

So since both colors sometimes happen to be red, the first statement of these three says that the shape of Bob's particle is never square when this happens. Likewise by the second of these statements, the shape of Alice's particle is also never square in the case that both particles are red. Hence when both particles are red, they both have to be round, but this contradicts the final statement!

This implication is known as **Hardy's nonlocality paradox** and due to [Har93]. The point is all four statements that (1) and (2) do occur with quantum correlations. In [Fri09b], I have argued that Hardy's nonlocality paradox should be viewed as the simplest **possibilistic Bell inequality**, i.e. a Bell inequality where one considers possibilities instead of probabilities. Just like ordinary Bell inequalities are facets of certain polytopes which can be calculated by Fourier-Motzkin eliminiation, possibilistic Bell inequalities can also be calculated by a Fourier-Motzkin-type algorithm. This algorithm can also be viewed as a variant of the resolution principle applied to propositional Horn formulae. It has been implemented as [Fri09a].

In fact, Hardy's nonlocality "paradox" is not a logical paradox. Like most pseudo-paradoxes, it contains a hidden assumption which is not satisfied in reality, making the occurence of Hardy's nonlocality possible and real [BBDMH97]. This assumption is that a quantity (shape or color in the present illustration) always has a definite value, even when not measured. This is the mystery of quantum correlations: observable quantities acquire definite values only due to their measurement!

**Brief recap of quantum kinematics.** A physical system is described by a Hilbert space $\mathcal{H}$. At each instant time, the system is in a certain **state** $|\psi\rangle$ which is a unit vector in $\mathcal{H}$. A measurement with $n$ different possible outcomes, is given by a complete family of mutually orthogonal projection operators $\Pi_k$,

$$\Pi_k : \mathcal{H} \to \mathcal{H}, \quad \Pi_k^* = \Pi_k, \quad \Pi_k \Pi_m = \delta_{km} \Pi_k, \quad \sum_k \Pi_k = \mathbb{1}, \quad k = 1, \ldots, n.$$

and the probability of getting the measurement outcome $k$ in some state $\Pi_k$ is given by the fundamental formula

$$\boxed{P(k) = \langle\psi|\Pi_k|\psi\rangle}$$

If the outcome $k$ occurs, then the system's state **collapses** to $P(k)^{-1/2}\Pi_k|\psi\rangle$ immediately. I.e. it gets projected onto the support of $\Pi_k$, and the factor $P(k)^{-1/2}$ then is merely the rescaling to a unit vector.

[1] The logical connective "$\diamond$" stems from modal logic and is to be interpreted as saying that its operand constitutes a possibility.

**The relevance of group $C^*$-algebras (mentioned in [Fri10]).** Given such a measurement **observable**, we can consider the operator

$$U \equiv \sum_{k=1}^{n} e^{2\pi i \frac{k}{n}} \Pi_k.$$

This is a unitary of order $n$, or, equivalently, a representation of the cyclic group $\mathbb{Z}_n$ on $\mathcal{H}$. Conversely, given a unitary of order $n$, we can consider its spectral decomposition, which gives a complete set of $n$ mutually orthogonal projection operators $\Pi_k$. This shows that an observable with $n$ outcomes is the same thing as a representation of $\mathbb{Z}_n$. In terms of the group $C^*$-algebra $C^*(\mathbb{Z}_n)$, an observable with $n$ outcomes is the same thing as a $*$-homomorphism

$$C^*(\mathbb{Z}_n) \to \mathcal{B}(\mathcal{H})$$

Now suppose that we have a physical system on which we can measure one of several observables—just like Alice and Bob can in Hardy's nonlocality paradox above. Let us consider $M$ observables having $n_1, \ldots, n_M$ outcomes, respectively. By the universal property of unital free products, a specification of these $M$ observables is equivalent to a $*$-homomorphism

$$C^*(\mathbb{Z}_{n_1}) * \ldots * C^*(\mathbb{Z}_{n_M}) \longrightarrow \mathcal{B}(\mathcal{H}).$$

In terms of maximal group $C^*$-algebras, we have an isomorphism

$$C^*(\mathbb{Z}_{n_1}) * \ldots * C^*(\mathbb{Z}_{n_m}) \cong C^*(\Gamma) \quad \text{with} \quad \Gamma = \mathbb{Z}_{n_1} * \ldots * \mathbb{Z}_{n_M}. \tag{3}$$

Hence, a specification of such observables is equivalent to a representation of $C^*(\Gamma)$. Furthermore, any state $|\psi\rangle$ in such a representation can be pulled back to a $C^*$-algebraic $\rho$ state on $C^*(\Gamma)$, and this $\rho$ encodes all the information about the quantum correlations. Conversely, given a $C^*$-algebraic state on $C^*(\Gamma)$, one can use the GNS construction to obtain a model of a physical system displaying the same behavior as $\rho$.

The main point now is that this allows a **universal quantification over all physical systems**, together with specifications for all measurements, at once. The question of existence of a certain type of quantum correlations then becomes a question about $C^*(\Gamma)$ and its state space.

**Example: results for two iterated binary measurements [Fri10].** A very simple case is when $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$. It is known that [RS89]

$$C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \left\{ f : [0,1] \stackrel{\text{cont.}}{\longrightarrow} M_2(\mathbb{C}) \;\middle|\; f(0),\, f(1) \text{ are diagonal} \right\}$$

This result can be used e.g. to give an easier proof than in [Mas05] for the fact that all nonlocal quantum correlations with two binary measurements per site can be modelled on qubits. Or for characterizing which kind of probabilities can appear on a quantum system without dynamics which can be subjected iteratively to two binary measurements $a$, $b$ [Fri10]. When $s$ is any binary string, one obtains two hierarchies of constraints as follows:

(a) $P_{aba\ldots}(0, s)$ and $P_{aba\ldots}(1, s)$ both depend only on the number of switches in the sequence $s$, i.e. the number of indices $k$ such that $s_k \neq s_{k+1}$.

(b) $P_{aba\ldots}(s) + P_{aba\ldots}(\overline{s}) = P_{bab\ldots}(s) + P_{bab\ldots}(\overline{s})$, where $\overline{s}$ is the "negation" of $s$, i.e. $\overline{s}$ is $s$ with $0 \leftrightarrow 1$ interchanged.

For example, the first instance of the first hierarchy states that $P_{aba}(0,0,1) = P_{aba}(0,1,1)$. So upon conditioning with respect to the two outcomes of $a$ being different, **any** quantum system with two binary measurements can in principle function as a perfectly unbiased random number generator!

**Outlook: From Kirchberg's conjectures to Tsirelson's problem.** This section should be considered as speculation and as an outline of plans for future work.

Usually when one considers quantum nonlocality scenarios like above, it is assumed that the Hilbert space of states of the whole system is a tensor product of the Hilbert spaces of states of the constituent systems:

$$\mathcal{H} = \mathcal{H}_{\mathrm{Alice}} \otimes \mathcal{H}_{\mathrm{Bob}}.$$

But alternatively, one can also define nonlocal quantum correlations by simply assuming that Alice's observables commute with Bob's observables in the Hilbert space of states of the total system, schematically:

$$\Pi_k^{\mathrm{Alice}} \Pi_l^{\mathrm{Bob}} = \Pi_l^{\mathrm{Bob}} \Pi_k^{\mathrm{Alice}}$$

The problem now is this:

> **Tsirelson's problem [SW08].** Do these two approaches give the same sets of quantum correlations?

It can now be shown that these two approaches respectively correspond to working with two different $C^*$-algebras:

$$C^*(\Gamma^{\mathrm{Alice}}) \otimes_{\min} C^*(\Gamma^{\mathrm{Bob}}) \quad \text{vs.} \quad C^*(\Gamma^{\mathrm{Alice}}) \otimes_{\max} C^*(\Gamma^{\mathrm{Bob}})$$

There are two properties which a particular $C^*$ may have or may not have [Kir93], [Oza04], [Pis03]:

(a) the weak expectation property (WEP),

(b) the local lifting property (LLP).

Kirchberg [Kir93] and others have shown the following facts:

> **Fact 1 [Oza04, 3.17].** If $A$ has the LLP and $B$ has the WEP, then $A \otimes_{\min} B = A \otimes_{\max} B$.

> **Fact 2 [Oza04, 3.21].** If $A_1$ has the LLP and $A_2$ has the LLP, then so has their free product $A_1 * A_2$.

The main point now is this:

> **QWEP conjecture (Kirchberg).** If $A$ has the LLP, then it has the WEP.

Using fact 2, it easily follows that the group $C^*$-algebras (3) do have the LLP. Hence from fact 1, it follows that

$$\textbf{QWEP conjecture} \quad \Longrightarrow \quad \begin{pmatrix} C^*(\Gamma^{\mathrm{Alice}}) \otimes_{\min} C^*(\Gamma^{\mathrm{Bob}}) \\ = C^*(\Gamma^{\mathrm{Alice}}) \otimes_{\max} C^*(\Gamma^{\mathrm{Bob}}) \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \textbf{solution to} \\ \textbf{Tsirelson's problem} \end{pmatrix}$$

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