Witnessing Infinite-dimensional State Spaces

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Motivation

- Standard quantum field theory predicts the Hilbert space of a quantum field to be infinite-dimensional.
- Quantum field theory is probably not the most fundamental description of reality.
- Approaches to a fundamental description may predict the Hilbert space of a quantum field to be of finite dimension.
- How does this change the predictions of the theory?
- For most purposes, an infinite-dimensional Hilbert space can be approximated by finite-dimensional ones.
  Example: $\mathcal{H} = \ell^2(\mathbb{N})$, state $\rho \in C_1(\mathcal{H})$, observable $A \in \mathcal{B}(\mathcal{H})$, projectors $P_n = \sum_{i=1}^{n} |e_i\rangle\langle e_i|$. Then

$$\operatorname{tr}\left(P_n \rho P_n \cdot P_n A P_n\right) \overset{n\to\infty}{\longrightarrow} \operatorname{tr}(\rho A) \,.$$

  $\Rightarrow$ Finite-dimensional approximability holds for expectation values of observables.

Main Theorem

Question

Does this finite-dimensional approximability hold in every physical context[1]?

Answer: No.

Theorem

(a) If $\dim(\mathcal{H}) < \infty$ and $U, V \in U(\mathcal{H})$, then

$$V^{-1}U^2V = U^3 \quad \Longrightarrow \quad UV^{-1}UV = V^{-1}UVU\,.$$

(b) If $\dim(\mathcal{H}) = \infty$, then there are $U, V \in U(\mathcal{H})$ and $\psi \in \mathcal{H}$ s.t.

$$V^{-1}U^2V = U^3\,, \quad \text{but} \quad \langle UV^{-1}UV\,\psi, V^{-1}UVU\,\psi\rangle = 0\,.$$

Proof (with heavy machinery).

Combine [2] with [4]. As a special case, the Baumslag-Solitar group $BS(2,3)$ is not maximally almost periodic. □

[1] I don't know how to rigorously define "physical context" in this context.

Outline of this talk

- Sketch a short elementary proof of the theorem which adapts to the consideration of experimental inaccuracies.
- Briefly discuss whether this result is suitable for detecting infinite-dimensionality
  - in principle,
  - in practice.

Part 1: Proving the Main Theorem

Theorem (part (a))

If $d = \dim(\mathcal{H}) < \infty$, then $V^{-1}U^2V = U^3 \;\Rightarrow\; UV^{-1}UV = V^{-1}UVU$.

Proof.

- By assumption, $\left(V^{-1}UV\right)^2 = U^3$ commutes with U.
- It needs to be shown that also $V^{-1}UV$ commutes with U.
- To this end, prove that U^2 does not have more degenaracy than U:

$$\textbf{Claim:} \quad \lambda \in \mathrm{spec}(U^2) \;\Rightarrow\; \exists!\, \lambda' \in \mathrm{spec}(U) \;\text{ with }\; \lambda = \lambda'^2 \,.$$

- We will show: if $\lambda_0 \in \mathrm{spec}(U)$, then $-\lambda_0 \notin \mathrm{spec}(U)$.
- Start by observing that by conjugacy of U^2 and U^3,

$$\lambda_0 \in \mathrm{spec}(U) \;\Rightarrow\; \exists \lambda_1 \in \mathrm{spec}(U) \;\text{ with }\; \lambda_0^3 = \lambda_1^2 \,.$$

- Iterating this gives a sequence $\lambda_0, \ldots, \lambda_d \in \mathrm{spec}(U)$ with

$$\lambda_n^3 = \lambda_{n+1}^2 \,.$$

Theorem (part (a))

If $d = \dim(\mathcal{H}) < \infty$, then $V^{-1}U^2V = U^3 \Rightarrow UV^{-1}UV = V^{-1}UVU$.

Proof, continued.

- By $|\mathrm{spec}(U)| \leq d$ and the pigeonhole principle, $\lambda_k = \lambda_{k+n}$ for some k and $n \geq 1$. This implies
$$\lambda_k^{3^n - 2^n} = 1\,.$$
- Therefore
$$\lambda_0^{3^k(3^n - 2^n)} = 1\,.$$
- Since $\lambda_0 \in \mathrm{spec}(U)$ was arbitrary, taking the l.c.m. of the exponent over all $k, n \leq d$ yields $N(d) \in \mathbb{N}$ such that $\lambda^{N(d)} = 1$ for all $\lambda \in \mathrm{spec}(U)$.
- Since $N(d)$ is odd, this cannot be satisfied by both λ and $-\lambda$ at the same time.

□

Theorem (part (b))

If $\dim(\mathcal{H}) = \infty$, then there are $U, V \in U(\mathcal{H})$ and $\psi \in \mathcal{H}$ such that $V^{-1}U^2V = U^3$ and $\langle UV^{-1}UV\,\psi, V^{-1}UVU\,\psi\rangle = 0$.

Proof.

- It is enough to show this for a particular separable $\mathcal{H}$.
- Define bijections $\mathbb{Z} \times \mathbb{N}_0 \xrightarrow{\cong} \mathbb{Z} \times \mathbb{N}_0$,

$$\begin{aligned} U &: (x,y) \mapsto (x+1, y) \\ V &: (x,y) \mapsto \left(\left\lfloor \frac{2x - (y \mod 2) + 1}{3} \right\rfloor, \left\lfloor \frac{3y + (x \mod 3)}{2} \right\rfloor \right) \end{aligned}$$

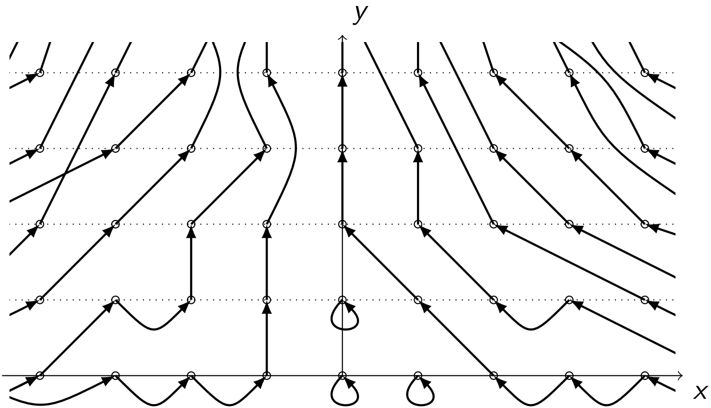
- Then $V^{-1}U^2V = U^3$

$$\begin{aligned} (UV^{-1}UV)(0,0) &= (0,2)\,, \\ (V^{-1}UVU)(0,0) &= (0,3)\,. \end{aligned}$$

- Linear extension to unitaries on $\ell^2(\mathbb{N}_0 \times \mathbb{Z})$.

□

U is a shift, while V acts as follows:



Taking inaccuracies into account

An actual experiment will not be able to verify the relation $V^{-1}U^2V = U^3$ exactly, but only up to finite error. To handle this, need a quantitative version of part (a) of the theorem:

Theorem

Let $d \geq 3$. If $\dim(\mathcal{H}) \leq d$ and $U, V \in U(\mathcal{H})$ satisfy

$$||V^{-1}U^2V - U^3|| < \varepsilon$$

for some $\varepsilon < \frac{1}{6 \cdot 3^{d(d+5)/2} d}$,then

$$||UV^{-1}UV - V^{-1}UVU|| < 4d^3 3^{d(d+3)/2} \varepsilon \, .$$

$\Rightarrow$ The witnessed dimension d scales with experimental accuracy ε^{-1} as

$$6 \cdot 3^{d(d+5)/2} d \sim \varepsilon^{-1} \, .$$

Part 2: Discussion (work in progress)

How to verify an equation between unitaries in an experiment?

Lemma

For transformations $W_1, W_2 \in U(\mathcal{H})$, the following two conditions are equivalent:

(a) $W_1 = e^{i\alpha} W_2$ for a phase $e^{i\alpha} \in U(1)$,

(b) For any initial state $\phi \in \mathcal{H}$ and measurement $A = A^* \in \mathcal{B}(\mathcal{H})$,

$$\langle W_1 \phi, A\, W_1 \phi \rangle = \langle W_2 \phi, A\, W_2 \phi \rangle \, . \tag{1}$$

Strategy for witnessing infinite-dimensions in an ideal experiment:

- Take a physical system for which standard quantum field theory predicts $\dim(\mathcal{H}) = \infty$,
- Find transformations $U, V \in U(\mathcal{H})$ with $V^{-1}U^2V = U^3$ and $\psi \in \mathcal{H}$ such that $\langle UV^{-1}UV\,\psi, V^{-1}UVU\,\psi\rangle = 0$. The existence is guaranteed by the theorem.
- Verify experimentally the relation $V^{-1}U^2V = U^3$ up to phase on as many initial states ϕ and observables A as possible.
- The phase is irrelevant since it can be absorbed by redefining U.
- Verify the relation $\langle UV^{-1}UV\,\psi, V^{-1}UVU\,\psi\rangle = 0$ by applying an appropriate measurement on the state $UV^{-1}UV\,\psi$ and comparing with the same measurement on the state $V^{-1}UVU\,\psi$.

$\Rightarrow$ If everything gives the expected results, the state space of the system is infinite-dimensional!

What happens in a real experiment?

- A real experiment has systematic and statistical errors.
- Therefore, the relation $V^{-1}U^2V = U^3$ cannot be verified perfectly, not even for a single initial state ϕ and observable A.

Two options:

(a) Use the second theorem to obtain a rigorous (but very weak) lower bound on d in terms of accuracy ε^{-1}.

(b) Argue by Occam's Razor: if the data suggests that the relation $V^{-1}U^2V = U^3$ holds up to experimental error, then any theory where this relation does not hold is unnatural.

- Another inaccuracy that will have to be taken into account: perfect unitaries are not realizable in practice.
  $\Rightarrow$ U and V have to be replaced by quantum channels (better: quantum operations).

Summary

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- I have described a context in which an infinite-dimensional Hilbert space cannot be approximated by finite-dimensional ones.
- Mathematically, the result is that the Baumslag-Solitar group $BS(2,3)$ is not maximally almost periodic.
- This follows from classical results in combinatorial group theory.
- We give an elementary and self-contained proof of this which adapts to the consideration of inaccuracies.
- $\Rightarrow$ One can witness infinite-dimensionality an in ideal experiment, while in a real experiment with inaccuracies one obtains a lower bound on the dimension.
- Due to bad scaling of this lower bound, looking for an experimental implementation would be pointless.
- Nevertheless, arguing by Occam's razor could witness infinite-dimensionality of the simplest model.
- Question: Are there variants of this proposal where the infinite-dimensionality witness survives the consideration of errors?

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