Resources

Tobias Fritz

joint work with Bob Coecke and Rob Spekkens

Perimeter Institute

May 2013

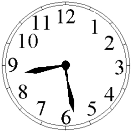
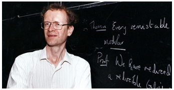
$$\xrightarrow{?}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

What are resources?

Resources are objects which can be converted into each other via processes:

$\rightsquigarrow$

 +  +  $\rightsquigarrow$ $\boxed{\begin{array}{c}\nexists x, y, z \in \mathbb{N},\\ x^n + y^n = z^n\end{array}}$

What are resources?

Resources are objects which can be converted into each other via processes:

+

A mathematical theory of resources

Definition

A **resource theory** is an ordered commutative monoid $(D, +, 0, \geq)$.

Explanation:

- D is the set of resources.
- Every two resources $a, b \in D$ can be combined into $a + b \in D$. This binary operation is associative and commutative.
- There is a resource $0 \in D$ which is trivial in the sense that $0 + a = a$.
- The relation $a \geq b$ stands for

  There is a process which turns a into b.
- If $a \geq b$ and $b \geq a$, then $a = b$. Interpretation: equivalent resources are represent by the same element of D.
- If $a \geq b$ and $a' \geq b'$, then

$$a + a' \geq b + b'.$$

Example: the resource theory of chemistry I

Let **Chem** be the ordered commutative monoid in which the resources $a, b, \ldots \in$ **Chem** are collections of molecules like

$$2H_2O_2, \qquad 2H_2O + O_2, \qquad \ldots$$

and $a \geq b$ if there is a chemical reaction (under standard conditions) of type $a \to b$.

Example: the resource theory of chemistry II

We have

$$2H_2O_2 \not\geq 2H_2O + O_2, \qquad (1)$$

but

$$2H_2O_2 + MnO_2 \geq 2H_2O + O_2 + MnO_2.$$

This is an example of **catalysis**, a phenomenon which may occur in any resource theory.

As one should expect from any mathematical model of real-world phenomena, the mathematical structure of **Chem** is highly idealized.

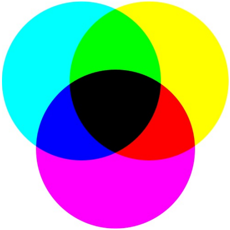
For example, equation (1) is, strictly speaking, false: there is a reaction $2H_2O_2 \to 2H_2O + O_2$ under standard conditions, but it is extremely slow.

Example: the resource theory of paints

Let **Paint** be the ordered commutative monoid in which a resource $a \in$ **Paint** is a collection of buckets, where each bucket in a contains paint of a certain color.

Collection of buckets can be joined, which is a binary operation $+$. The empty collection of buckets is 0.

The paints in different buckets can be mixed:



$a \geq b$ for $a, b \in$ **Paint** if b can be obtained from a by mixing paints and/or discarding buckets.

Types of resources I

Definition

$a \in D$ is **disposable** if $a \geq 0$.

Many resources are not disposable:

These "resources" are undesirable: getting rid of them is costly or even impossible, hence their production should be avoided.

Types of resources II

Definition

$m \in D$ is a **machine** if $2m + a \geq b$ implies $m + a \geq b$.

Idea: having one machine is as good as having two units of it.

If m is a machine, then $km + a \geq b$ for any $k \in \mathbb{N}$ implies $m + a \geq b$.
For example, $0 \in D$ trivially is a machine.

Types of resources III

Definition

$\$ \in D$ is a **currency** if for all $a \in D$, there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that

$$\forall \varepsilon > 0 \quad \exists n, k, m \in \mathbb{N}, \quad na \geq k\$ \geq ma,$$

$$\frac{k}{n} \in (\lambda - \varepsilon, \lambda + \varepsilon), \quad \frac{m}{n} \in (1 - \varepsilon, 1 + \varepsilon).$$

Idea: instances of a can be “sold” in exchange for (roughly) $\lambda\,\$$, which can then be used to “buy” other resources having their own prices.

Types of resources IV

Definition

A resource $u \in D$ is **universal** if for every $a, b \in D$ there exists $n \in \mathbb{N}$ such that

$$nu + a \geq b.$$

Idea: A universal resource can be used to turn any resource a into any resource b.

In most resource theories of interest, a universal resource does exist.

Example: in **Paint**, take one bucket of each primary color.

$\longrightarrow$ Existence of a universal resource is a technical assumption in some theorems.

Types of resource theories I

Definition

D is **catalysis-free** if

$$a + c \geq b + c \quad \Longrightarrow \quad a \geq b.$$

Definition

D is **non-interacting** if

$$a \geq b_1 + b_2 \quad \Longrightarrow \quad \exists a_1, a_2 \in D,\ a = a_1 + a_2,\ a_1 \geq b_1,\ a_2 \geq b_2.$$

Idea: every process which outputs a combination of two resources can be decomposed into parallel application of two processes each of which outputs a constituent resource.

Types of resource theories II

Definition

D is **complimentary** if

$$a + a' = b + b', \;\; a \geq b \quad \Longrightarrow \quad b' \geq a'.$$

Proposition

If D is non-interacting and complimentary, then D is catalysis-free.

Proof.

Assume that $a + c \geq b + c$. Since D is non-interacting, find $a + c = a_1 + a_2$ with $a_1 \geq b$ and $a_2 \geq c$. Since D is complimentary, $a + c = a_1 + a_2$ and $a_2 \geq c$ implies $a \geq a_1$. From $a \geq a_1 \geq b$ we conclude $a \geq b$. □

Rates I

In many situations, we would like to produce many units of a resource b from many units of a resource a.

Mass production increases efficiency!

$\longrightarrow$ In the asymptotic limit, how many units of b can be produced per unit of a?

Rates II

Let u be a universal resource.

Definition

Let $a, b \in D$. A given $\lambda \in \mathbb{R}_{\geq 0}$ is a **rate** from a to b if

$$\forall \varepsilon > 0 \quad \exists n, k, m \in \mathbb{N}, \quad na + ku \geq mb,$$

$$\tfrac{m}{n} \in (\lambda - \varepsilon, \lambda + \varepsilon), \quad \tfrac{k}{n} < \varepsilon.$$

Note: this has nothing to do with **reaction rates** in chemistry! The rates in the resource theory **Chem** are the subject of **stoichiometry**.

Proposition

For any D and any $a, b \in D$, the set of all rates is an interval $[R_{\min}(a \to b), R_{\max}(a \to b)]$.

Rates III

Definition

A **valuation** is a function $V : D \to \mathbb{R}$ such that

- $V(a + b) = V(a) + V(b)$,
- if $a \geq b$, then $V(a) \geq V(b)$.

Idea: a valuation measures the value of resources in a consistent way.

Theorem (Fundamental Theorem of Rates)

If D has a universal pair of resources and b is disposable, then

$$R_{\min}(a \to b) = 0, \qquad R_{\max}(a \to b) = \inf_V \frac{V(a)}{V(b)}.$$

Epsilonification

In many resource theories, one does not require to outcome of a process to coincide *exactly* with a desired resource b; rather, it is enough if arbitrarily good approximations to b can be produced.

Examples:

- Resource theory of (quantum) communication channels,
- Resource theory of quantum entanglement,
- Resource theory of thermodynamics,
- ...

How do we know what constitutes an "approximation" to b?

The "right" answer to this is that the resource theory needs to be equipped with the mathematical structure of a **uniform space**.

The following definition treats the special situation when the uniform structure comes from a **metric** (measure of distance).

Epsilonification II

Definition (Tentative)

An **epsilonification** of a resource theory $(D, +, 0, \geq)$ is a metric $d : D \times D \to \overline{\mathbb{R}}_{\geq 0}$ satisfying the following additional conditions:

1. For every $\varepsilon > 0$, there is $\delta > 0$ such that
$$a \overset{\delta}{\leftrightsquigarrow} b, \qquad a' \overset{\delta}{\leftrightsquigarrow} b' \qquad \Longrightarrow \qquad a + a' \overset{\varepsilon}{\leftrightsquigarrow} b + b'.$$
2. For every $\varepsilon > 0$, there is $\delta > 0$ such that
$$a \overset{\delta}{\leftrightsquigarrow} a' \geq b' \overset{\delta}{\leftrightsquigarrow} b \qquad \Longrightarrow \qquad \begin{cases} \exists \hat{a},\ \ a \overset{\varepsilon}{\leftrightsquigarrow} \hat{a} \geq b \\ \exists \hat{b},\ \ a \geq \hat{b} \overset{\varepsilon}{\leftrightsquigarrow} b \end{cases} .$$
3. If $d(na, nb) \to 0$ for $n \to \infty$, then $a = b$.

Here, $a \overset{\varepsilon}{\leftrightsquigarrow} b$ stands for $d(a, b) < \varepsilon$.

Epsilonification III

- The mathematical theory of epsilonified resource theories is an interesting blend of algebra, order theory, and analysis.

  For example:

Proposition

The monoid $(D, +, 0)$ is torsion-free: if $a, b \in D$ and $n \in \mathbb{N}$ such that $na = nb$, then $a = b$.

Proof.

Immediate from Axiom 3. $\square$

- Most concepts and theorems for resource theories have analogues in the epsilonified case. (Work in progress.)
- For example, a valuation also is assumed to be uniformly continuous.

The resource theory of randomness I

- A resource in **Rand** is a finite sequence $p = (p_1, \ldots, p_n)$ with
$$p_1 \geq p_2 \geq \ldots \geq p_n \geq 0 \quad \text{and} \quad \sum_k p_k = 1.$$
Equivalently: an isomorphism class of finite probability spaces.
- The combination of resources $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_m)$ in **Rand** is given by taking product distributions:
$$p + q \stackrel{\mathrm{def}}{=} (p_1 q_1, \ldots, p_1 q_m, \ldots, p_n q_1, \ldots, p_n q_m),$$
suitably reordered.
- $0 \in$ **Rand** is the deterministic distribution (1).
- We put $p \geq q$ whenever q is a **coarse-graining** of p: there is a function
$$f : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$$
such that $q_k = \sum_{j \in f^{-1}(k)} p_j$.

The resource theory of randomness II

- The distance between $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_m)$ is

$$d(p,q) \stackrel{\text{def}}{=} \begin{cases} \infty & \text{if } n \neq m \\ \sum_k |p_k - q_k| & \text{if } n = m \end{cases}.$$

- With this, **Rand** is an epsilonified resource theory.
- **Rand** is of fundamental importance in **information theory**.

Definition

The **partition function** of p is

$$Z_p(\beta) = \sum_k p_k^\beta.$$

where $\beta \in [0, \infty]$. The **Rényi entropy** of p of order β is

$$H_\beta(p) = \tfrac{1}{1-\beta} \log Z_p(\beta).$$

The resource theory of randomness III

For $\beta \to 1$, one obtains the **Shannon entropy**

$$H_1(p) = -\sum_k p_k \log p_k .$$

Proposition

The Rényi entropy H_β : **Rand** $\to \mathbb{R}$ is a continuous valuation if and only if $\beta > 1$.

Corollary (Tentative)

$$R_{\max}(p \to q) \leq \min_{\beta \in [1,\infty]} \frac{H_\beta(p)}{H_\beta(q)}$$

Conjecture: this holds with equality.

$\longrightarrow$ Operational meaning for Rényi entropy!

Thanks

Bob Coecke

Rob Spekkens

John Baez

Cecilia Flori

Markus Müller