PMATH 911: Assignment 3

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Question 1

Part a

We proceed by induction on x.
For $x \equiv 0$, we need a term of type $(0 = 0) + (0 = 1)$. We have $\mathrm{refl}_0 : 0 = 0$, so we can take $\mathrm{inl}(\mathrm{refl}_0)$.
Similarly for $x \equiv 1$, we can take $\mathrm{inr}(\mathrm{refl}_1)$. $\square$

Part b

The uniqueness principle is as follows:

$$\prod_{n:\mathbb{N}} \left((n = 0) + \sum_{m:\mathbb{N}} (n = \mathrm{succ}(m)) \right)$$

To prove this, we proceed by induction on n.
For $n \equiv 0$, we need a term of type $(0 = 0) + \sum_{m:\mathbb{N}}(0 = \mathrm{succ}(m))$. We have $\mathrm{refl}_0 : 0 = 0$, so we can take $\mathrm{inl}(\mathrm{refl}_0)$.
For $n \equiv \mathrm{succ}(k)$ and $p : (k = 0) + \sum_{m:\mathbb{N}}(k = \mathrm{succ}(m))$ (which we do not need), we need a term of type $(\mathrm{succ}(k) = 0) + \sum_{m:\mathbb{N}}(\mathrm{succ}(k) = \mathrm{succ}(m))$. We have $(k, \mathrm{refl}_{\mathrm{succ}(k)}) : \sum_{m:\,\mathbb{N}}(\mathrm{succ}(k) = \mathrm{succ}(m))$, so we can take $\mathrm{inr}((k, \mathrm{refl}_{\mathrm{succ}(k)}))$. $\square$

Question 2

Part a

We need to prove the following:

$$\prod_{f,g:A\to\mathbf{1}} \prod_{x:A} (f(x) = g(x))$$

If we fix f, g, x, then we just need a term of type $f(x) = g(x)$. Recall that from the uniqueness principle of the unit type, there exist $p : f(x) = *$ and $q : g(x) = *$, so we are done since $=$ is an equivalence relation (explicitly we can take $p^{-1} \cdot q : f(x) = g(x)$). $\square$

Part b

Let $A \equiv \texttt{Bool}$ and let $f \equiv \lambda x.0$ and $g \equiv \lambda x.1$. Then we need a term of type $(f \sim g) \to \mathbf{0}$. Note that $f(*) \equiv 0$ and $g(*) \equiv 1$. Also recall that from class, we have $\phi : (0 = 1) \to \mathbf{0}$. Using these facts, we can take $h : (f \sim g) \to \mathbf{0}$ defined by $h \equiv \lambda p.\phi(p(*))$ and we are done. $\square$

Question 3

We need a term of the following type:

$$\prod_{x,y,z:A} \prod_{p,q:x=y} \prod_{r:y=z} \prod_{\alpha:p=q} \alpha \diamond r = \mathrm{ap}_{\_\cdot r}(\alpha)$$

By path induction on p, we can take $y \equiv x$ and $p \equiv \mathrm{refl}_x$. Then we have:

$$\prod_{x,z:A} \prod_{q:x=x} \prod_{r:x=z} \prod_{\alpha:\mathrm{refl}_x=q} \alpha \diamond r = \mathrm{ap}_{\_\cdot r}(\alpha)$$

By path induction on α, we can take $q \equiv \mathrm{refl}_x$ and $\alpha \equiv \mathrm{refl}_{\mathrm{refl}_x}$. We can also path induct on r and take $z \equiv x$ and $r \equiv \mathrm{refl}_x$. Then we have:

$$\prod_{x:A} \mathrm{refl}_{\mathrm{refl}_x} \diamond \mathrm{refl}_x = \mathrm{ap}_{\_\cdot \mathrm{refl}_x}(\mathrm{refl}_{\mathrm{refl}_x})$$

By the computation rule for whiskering, we have:

$$\mathrm{refl}_{\mathrm{refl}_x} \diamond \mathrm{refl}_x \equiv \mathrm{refl}_{\mathrm{refl}_x}$$

By the computation rules for ap and composition, we have:

$$\mathrm{ap}_{\_\cdot \mathrm{refl}_x}(\mathrm{refl}_{\mathrm{refl}_x}) \equiv \mathrm{refl}_{\mathrm{refl}_x \cdot \mathrm{refl}_x} \equiv \mathrm{refl}_{\mathrm{refl}_x}$$

So a term of the desired type is just $\mathrm{refl}_{\mathrm{refl}_{\mathrm{refl}_x}}$, and we are done. □

Question 4

We will show that for any $f : \prod_{n:\mathbb{N}} C(n)$ satisfying recursion with respect to b and s, we have that $f \sim \mathrm{ind}_{\mathbb{N}}(C, b, s)$, and the result follows since homotopy is an equivalence relation. Since f satisfies recursion, we have a term ϕ of the following type:

$$\phi \colon (f(0) = b) \times \prod_{n:\mathbb{N}} f(\mathrm{succ}(n)) = s(n, f(n))$$

The type $f \sim \mathrm{ind}_{\mathbb{N}}(C, b, s)$ is defined as the following type:

$$\prod_{n:\mathbb{N}} (f(n) = \mathrm{ind}_{\mathbb{N}}(C, b, s, n))$$

To prove this, we proceed by induction on n. If $n \equiv 0$, we need a term of type $f(0) = \mathrm{ind}_{\mathbb{N}}(C, b, s, 0)$. Note that $\mathrm{ind}_{\mathbb{N}}(C, b, s, 0) \equiv b$, so we just need a term of type $f(0) = b$. For this, we can take $\mathrm{pr}_1(\phi)$.
For $n \equiv \mathrm{succ}(k)$ and $p : f(k) = \mathrm{ind}_{\mathbb{N}}(C, b, s, k)$, we need a term of type $f(\mathrm{succ}(k)) = \mathrm{ind}_{\mathbb{N}}(C, b, s, \mathrm{succ}(k))$.
Note that $\mathrm{ind}_{\mathbb{N}}(C, b, s, \mathrm{succ}(k)) \equiv s(k, \mathrm{ind}_{\mathbb{N}}(C, b, s, k))$, so we just need a term of type $f(\mathrm{succ}(k)) = s(k, \mathrm{ind}_{\mathbb{N}}(C, b, s, k))$.
We have $q \equiv \mathrm{pr}_2(\phi)(k) : f(\mathrm{succ}(k)) = s(k, f(k))$. Note also that $\mathrm{ap}_{s(k,\_)}(p) : s(k, f(k)) = s(k, \mathrm{ind}_{\mathbb{N}}(C, b, s, k))$.
Thus our desired path is $q \cdot \mathrm{ap}_{s(k,\_)}(p)$. □

Question 5

Before we do the question, note that for any $A : \mathcal{U}$, we always have $\mathrm{ind}_{\mathbf{0}}(A) : \mathbf{0} \to A$.
Now, suppose we have $C : \mathbb{N} \to \mathcal{U}$ and we also have f of the following type:

$$f : \prod_{n:\mathbb{N}} \left(\prod_{k:\mathbb{N}} (k < n) \to C(k) \right) \to C(n)$$

We wish to prove the following:

$$\prod_{n:\mathbb{N}} C(n)$$

Thus it suffices to find a term ϕ of the following type:

$$\phi : \prod_{n:\mathbb{N}} \prod_{k:\mathbb{N}} (k < n) \to C(k)$$

and then we can just take $\lambda n. f(n, \phi(n)) : \prod_{n:\mathbb{N}} C(n)$.
To find ϕ, we proceed by induction on n. We will also fix $k : \mathbb{N}$.
For $n \equiv 0$, we need $(k < 0) \to C(k)$. But since $(k < 0) \equiv \mathbf{0}$, by the first sentence there exists a term of type $\mathbf{0} \to C(k)$ and we are done.
When $n \equiv \mathrm{succ}(m)$ and we have $g : (k < m) \to C(k)$, we need to find a term of type $(k < \mathrm{succ}(m)) \to C(k)$. Note that $(k < \mathrm{succ}(m)) \equiv (k < m) + (k = m)$. Thus to find a term of type $(k < \mathrm{succ}(m)) \to C(k)$, it suffices to find a term of type $(k < m) \to C(k)$ and $(k = m) \to C(k)$. We already have $g : (k < m) \to C(k)$, so remains to find a term of type $(k = m) \to C(k)$. First of all, note that we have $f(m, g) : C(m)$. Thus using transport, we get $\lambda p. \mathrm{transport}^C(p^{-1}, f(m, g)) : (k = m) \to C(k)$. □