On this assignment we will score the best four of five solutions, for a maximum of 40 points.

If you do the exercises in `Coq`, then you should use the HoTT version at

https://github.com/HoTT/HoTT

from now on. Let us know if you need help with installing this or instructions on how to use it. The HoTT library containing the standard functions (path concatenation, transport etc.) is located in the subdirectory `./theories`.

1. **More propositional uniqueness principles.** (10 points)
   (a) Show that the Booleans satisfy
   $$\prod_{x:\mathtt{Bool}} (x = 0) + (x = 1).$$
   (b) Formulate and prove a uniqueness principle for the natural numbers.

2. **Homotopy.** (10 points)
   (a) Show that any two functions $f, g : A \to \mathbf{1}$ are homotopic for any $A : \mathcal{U}$.
   (b) Find $A : \mathcal{U}$ and $f, g : \mathbf{1} \to A$ such that $f$ and $g$ are not homotopic, i.e. $(f \sim g) \to \mathbf{0}$.

3. **Another point of view on whiskering.** (10 points)
   For $A : \mathcal{U}$ a type with elements $x, y, z : A$ and $p, q : x = y$, show that
   $$\prod_{r:y=z} \prod_{\alpha:p=q} \alpha \bullet r = \mathrm{ap}_{\_\cdot r}(\alpha),$$
   where we write $\_\cdot r$ for concatenation with $r$, i.e. $\_\cdot r :\equiv \lambda(s : x = y).s \cdot r$.

4. **Uniqueness of inductively defined functions.** (10 points)
   For $C : \mathbb{N} \to \mathcal{U}$ a type family and $b : C(0)$ and $s : \prod_{n:\mathbb{N}} C(n) \to C(\mathtt{succ}(n))$, we say that a dependent function $f : \prod_{n:\mathbb{N}} C(n)$ satisfies recursion with respect to $b$ and $s$ whenver
   $$(f(0) = b) \times \prod_{n:\mathbb{N}} f(\mathtt{succ}(n)) = s(n, f(n)).$$
   (This is a propositional version of the computation rule that one obtains if one defines $f$ by induction.) Show that if two functions $f, g : \prod_{n:\mathbb{N}} C(n)$ both satisfy recursion with respect to $b$ and $s$, then $f \sim g$.

5. **Strong induction.** (10 points)
   For $x, y : \mathbb{N}$, define the type $(x < y) : \mathcal{U}$ by induction on $y$,
   $$(x < 0) :\equiv \mathbf{0}, \qquad (x < \mathtt{succ}(y)) :\equiv (x < y) + (x = y).$$
   Derive the *principle of strong induction*, which is the statement
   $$\prod_{C:\mathbb{N}\to\mathcal{U}} \left( \prod_{m:\mathbb{N}} \left( \prod_{k:\mathbb{N}} (k < m) \to C(k) \right) \to C(m) \right) \to \prod_{n:\mathbb{N}} C(n).$$
   Intuitive explanation: in order to construct a sequence of elements $a_n : C(n)$, it is enough to define each $a_m$ as a function of the $a_k$ with $k < m$.

   *Hint:* It may help to know how to derive this in conventional foundations. See for example http://mathoverflow.net/questions/37944/induction-vs-strong-induction.