Measurement Functors

Based on arXiv:1512.01669

March 2018

- We often talk about the poset of commutative subalgebras of the algebra of observables A.
- This does not have nice functoriality properties in A.
  What is the reason for using commutative *sub*algebras anyway?
- A better solution is to consider *all* $*$-homomorphisms $C \to A$ for *all* commutative C.
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- We often talk about the poset of commutative subalgebras of the algebra of observables A.
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  What is the reason for using commutative *sub*algebras anyway?
- A better solution is to consider *all* $*$-homomorphisms $C \to A$ for *all* commutative C.
  This is nicely functorial in A.
- In a C*-setting, this means that we consider all $*$-homomorphisms $C(X) \to A$ for all $X \in \mathrm{CHaus}$.
- So we associate to every A the functor

$$\mathrm{CHaus} \to \mathrm{Set}, \qquad X \mapsto \mathrm{C^*alg}_1(C(X), A).$$

- By Gelfand duality, this is equivalent the restricted Yoneda embedding $\mathrm{C^*alg}_1 \to \mathrm{Set}^{\mathrm{cC^*alg}_1^{\mathrm{op}}}$.

- Generally, we can start with a physical system in any theoretical framework.
- For every space $X \in \mathrm{CHaus}$, there should be defined a set $M(X)$, namely the set of all possible measurements with outcomes in X.
- For every $f : X \to Y$ in CHaus, there should be defined a function

$$M(f) : M(X) \to M(Y)$$

which implements the idea of post-processing along f.
- Thus we obtain a functor $M : \mathrm{CHaus} \to \mathrm{Set}$, the **measurement functor** describing the system.

Question

How much information about the system is contained in its measurement functor?

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- Consider the case of a quantum system described by a C*-algebra A.
- Then we write
$$X(A) := \mathrm{C^*alg}_1(C(X), A)$$
for the value of the measurement functor associated to A on X.
- Notation in analogy with algebraic geometry: for X a scheme and A a commutative ring, $X(A)$ is the set of points X over A.
- Our idea is the same, except: now A is fixed rather than X.
- For example, $[-1,+1](A)$ is the set of self-adjoints $x \in A$ with $\|x\| \leq 1$. By scaling, we reconstruct all self-adjoints in A together with their norm!

- More generally, for every compact $X \subseteq \mathbb{C}$, we can identify $X(A)$ with the set of normal elements with spectrum in X.

- Applying an $f : (X \subseteq \mathbb{C}) \to (Y \subseteq \mathbb{C})$ is the usual functional calculus.

- Thus, our measurement functor $-(A) : \mathsf{CHaus} \to \mathsf{Set}$ generalizes functional calculus.

- In this spirit, we may also think of every $f \in X(A)$, represented by $f : C(X) \to A$, as a "generalized normal element".

Definition

The **category of measurement functors** is the functor cat $\mathsf{Set}^{\mathsf{CHaus}}$.

Question

How does $\mathsf{C^*alg_1}$ relate to $\mathsf{Set}^{\mathsf{CHaus}}$?

Definition (van den Berg & Heunen '10)

A **piecewise C*-algebra** is a set A equipped with

- a reflexive and symmetric relation $\perp\!\!\!\perp \subseteq A \times A$. If $\alpha \perp\!\!\!\perp \beta$, we say that α and β **commute**;
- binary operations $+, \cdot : \perp\!\!\!\perp \to A$;
- a scalar multiplication $\cdot : \mathbb{C} \times A \to A$;
- distinguished elements $0, 1 \in A$;
- an involution $* : A \to A$;
- a norm $|| - || : A \to \mathbb{R}$;

such that every $C \subseteq A$ of pairwise commuting elements is contained in some $\bar{C} \subseteq A$ which is a commutative C*-algebra.

- Example: the normal elements of any C*-algebra form a piecewise C*-algebra.

- We have the category of piecewise C*-algebras $\mathsf{pC^*alg}_1$.
- We can still associate to every $A \in \mathsf{pC^*alg}_1$ a measurement functor,

$$\mathsf{CHaus} \to \mathsf{Set}, \qquad X \mapsto \mathsf{pC^*alg}_1(C(X), A).$$

  Thus we get $\mathsf{pC^*alg}_1 \to \mathsf{Set}^{\mathsf{CHaus}}$.
- By Gelfand duality, $\mathsf{CHaus}^{\mathrm{op}}$ is again a full subcategory, so that this is equivalent to the restricted Yoneda embedding

$$\mathsf{pC^*alg}_1 \to \mathsf{Set}_1^{\mathsf{cC^*alg}}.$$

Proposition

The functor $\mathsf{pC^*alg}_1 \to \mathsf{Set}^{\mathsf{CHaus}}$ is fully faithful.

The proof uses the fact that $x, y \in \bigcirc(A)$ commute if and only if they are in the image of

$$(\bigcirc \times \bigcirc)(A) \longrightarrow \bigcirc(A) \times \bigcirc(A).$$

- Thus we try to understand the essential image of $\mathsf{pC^*alg}_1 \to \mathsf{Set}^{\mathsf{CHaus}}$.
- Doing so results in a characterization and reconstruction of piecewise C*-algebras in terms of measurement functors.
- To this end, we will formulate a certain sheaf condition in several steps.

Definition

- A **cone** is a collection of morphisms $\{f_i : X \to Y_i\}_{i \in I}$ for some index set I.
- A cone is **effective-monic** if the diagram

$$X \longrightarrow \prod_{i \in I} Y_i \rightrightarrows \prod_{i,j \in I} (Y_i \; {}_{f_i}\amalg_{f_j} \; Y_j),$$

is an equalizer.

Example (Isbell '60, essentially)

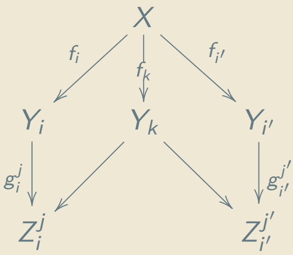
- For every $X \in \mathrm{CHaus}$, the cone of all functions $\{X \to \square\}$ is effective-monic, where $\square := [0,1]^2$.
- The same is not true with $[0,1]$ in place of $\square$.

- Equivalently: points of X are in bijection with **valuations** $f \mapsto \nu(f)$ operating on functions $f : X \to \square$, which are consistent in the sense that $\nu(g \circ f) = g(\nu(f))$ for all $g : \square \to \square$.

We also need a very technical additional condition:

Definition

An effective-monic cone $\{f_i : X \to Y_i\}_{i \in I}$ in CHaus is **directed** if for every $i \in I$ there is a cone $\{g_i^j : Y_i \to Z_i^j\}_{j \in J_i}$ which separates points, and such that for every $i, i' \in I$ and $j \in J_i$, $j' \in J_{i'}$ there is $k \in I$ and a diagram



- The effective-monic cone of all functions $\{X \to \square\}$ is directed.

Definition

A measurement functor $M \in \mathrm{Set}^{\mathrm{CHaus}}$ is a **sheaf** if and only if for every directed effective-monic cone $\{f_i : X \to Y_i\}_{i \in I}$, also

$$M(X) \longrightarrow \prod_{i \in I} M(Y_i) \rightrightarrows \prod_{i,j \in I} M(Y_i \; {}_{f_i}\amalg_{f_j} \; Y_j),$$

is an equalizer.

- The sheaf condition on $\{X \to \square\}$ "explains" why measurements in the lab can be assumed to be (complex) numerical.

Theorem

The essential image of $\mathrm{cC^*alg_1} \to \mathrm{Set}^{\mathrm{CHaus}}$ consists of those measurement functors which satisfy the sheaf condition on **all** effective-monic cones.

We write $\mathrm{Sh}(\mathsf{CHaus}) \subseteq \mathsf{Set}^{\mathsf{CHaus}}$ for the full subcategory of measurement functors satisfying the sheaf condition on all directed effective-monic cones.

Theorem

- The measurement functor associated to every piecewise C*-algebra is a sheaf.
- The resulting functor $\mathsf{pC^*alg_1} \to \mathrm{Sh}(\mathsf{CHaus})$ is fully faithful, with essential image given by those M for which

$$M(\square) \to M(\square) \times M(\square)$$

is injective.

- The injectivity condition says: for every two □-valued measurements, there is *at most one joint* measurement combining them.
- Open problem: is this condition necessary or automatically satisfied?

Thus we can answer the question:

Question

How much information about the system is contained in its measurement functor?

The answer is: **measurement functors satisfying the sheaf condition are the same thing as piecewise C*-algebras!**

- However, piecewise C*-algebras only capture the commutative aspects of C*-algebra theory.
- In particular, we cannot reconstruct the multiplication of noncommuting elements, and not even the addition!
- From the physical perspective, what is missing is **dynamics**: for every $h = h^* \in A$,

$$a \mapsto e^{-ith}\, a\, e^{ith}$$

  is a 1-parameter group of inner automorphisms of A.

- From the physical perspective, what is missing is **dynamics**: for every observable $h = h^* \in A$,

$$a \longmapsto e^{-ith}\, a\, e^{ith}$$

  is a 1-parameter group of inner automorphisms of A.

- This is one of the central features of quantum physics!

- Its construction proceeds in two steps:
  - exponentiate h. Being functional calculus, this is captured by M.
  - conjugating by the resulting unitary. This is not captured by M!

- Hence we axiomatize the action of inner automorphisms **as an extra piece of structure**.

Definition

An **almost C*-algebra** is an injective measurement sheaf $M : \mathsf{CHaus} \to \mathsf{Set}$ together with a **self-action**, which is a map

$$\mathfrak{a} : M(S^1) \longrightarrow \mathrm{Aut}(M)$$

such that if $u, v \in M(S^1)$ are jointly measurable, then

- $\mathfrak{a}(u)(v) = v$,
- $\mathfrak{a}(uv) = \mathfrak{a}(u)\mathfrak{a}(v)$.

- Here, it no longer matters whether we work with piecewise C*-algebras or injective measurement sheaves.
- The first equation is related to Noether's theorem.
- Every C*-algebra carries the structure of an almost C*-algebra.

Problem

Is the category of almost C*-algebras equivalent to the category of C*-algebras?

This question has two parts:

- Is every almost C*-algebra is isomorphic to a C*-algebra? This is wide open.
- For $A, B \in \mathrm{C^*alg}_1$, is every almost $*$-homomorphism $A \to B$ already a $*$-homomorphism? Here, we know:

Theorem

If A is a von Neumann algebra, then every almost $*$-homomorphism $A \to B$ is a $*$-homomorphism.

Problem

Is the category of almost C*-algebras equivalent to the category of C*-algebras?

- If the answer is positive, we have axioms for C*-algebras with clearer physical meaning.
- In particular, we would have the first reconstruction of *infinite-dimensional* quantum theory from (more) physical axioms.
- If the answer is negative, we can try to develop physical theories in terms of almost C*-algebras as alternatives to existing theories formulated in terms of C*-algebras. Could these be physically realistic? (Almost certainly not.)

Summary of proposed reconstruction

Roughly speaking, we have two kinds of axioms.

Measurements:

- Associated to every compact Hausdorff space X there is a set $M(X)$, comprising all measurements on the system with outcomes in X.
- Associated to every continuous function $f : X \to Y$, there is a *post-processing* or *coarse-graining* function $M(f) : M(X) \to M(Y)$.

Dynamics and Symmetry:

- Associated to every unitary u is an automorphism $\mathfrak{a}(u)$, satisfying suitable conditions.

In combination with the measurements structure, this results in:
associated to every observable is a 1-parameter family of automorphisms.

$\to$ Time evolution and other symmetries in physics.