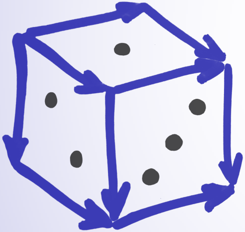
What is a probability monad?



Paolo Perrone

Massachusetts Institute
of Technology (MIT)

Categorical Probability 2020
Tutorial video

Monads as extensions

Definition:

Let C be a category. A *monad on* C consists of:

- A functor $T : \mathsf{C} \to \mathsf{C}$;
- A natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow T$ called *unit*;
- A natural transformation $\mu : TT \Rightarrow T$ called *composition*;

such that the following diagrams commute:

$$\begin{array}{ccc} T & \xRightarrow{T\eta} & TT \\ & {\scriptstyle \mathrm{id}} \searrow & \Downarrow \mu \\ & & T \end{array} \qquad \begin{array}{ccc} T & \xRightarrow{\eta T} & TT \\ & {\scriptstyle \mathrm{id}} \searrow & \Downarrow \mu \\ & & T \end{array} \qquad \begin{array}{ccc} TTT & \xRightarrow{T\mu} & TT \\ \Downarrow \mu T & & \Downarrow \mu \\ TT & \xRightarrow{\mu} & T \end{array}$$

Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

1. To each space X, an "extended" space TX.

Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

1. To each space X, an "extended" space TX.
2. Given $f : X \to Y$, an "extension" $Tf : TX \to TY$.

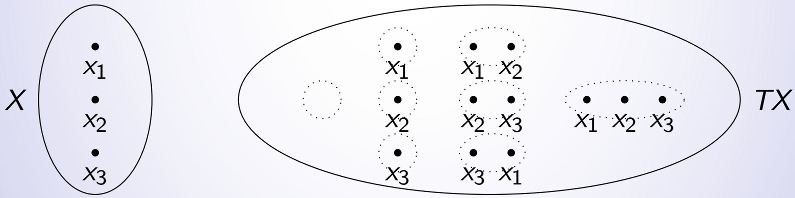
Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

1. To each space X, an "extended" space TX.
2. Given $f : X \to Y$, an "extension" $Tf : TX \to TY$.



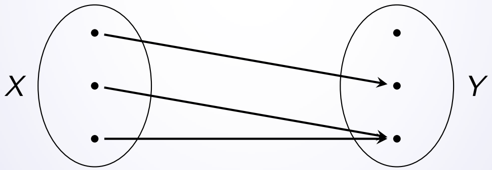
Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

1. To each space X, an "extended" space TX.
2. Given $f : X \to Y$, an "extension" $Tf : TX \to TY$.



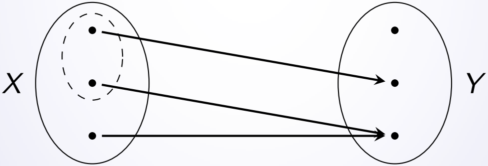
Monads as extensions

Idea:

A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

1. To each space X, an "extended" space TX.
2. Given $f : X \to Y$, an "extension" $Tf : TX \to TY$.



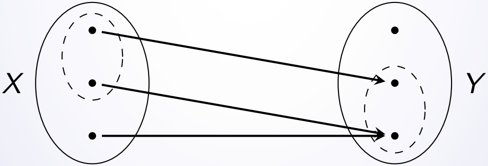
Monads as extensions

Idea:

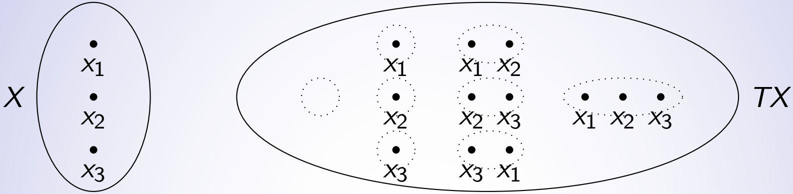
A monad is like a consistent way of extending spaces to include generalized elements and generalized functions of a specific kind.

A functor $T : \mathsf{C} \to \mathsf{C}$ consists of:

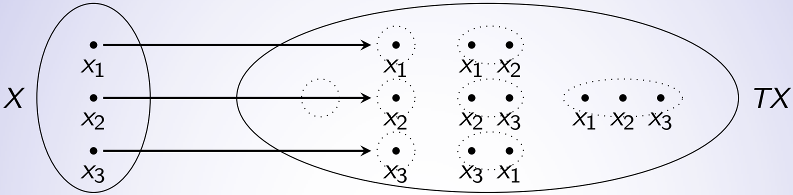
1. To each space X, an "extended" space TX.
2. Given $f : X \to Y$, an "extension" $Tf : TX \to TY$.



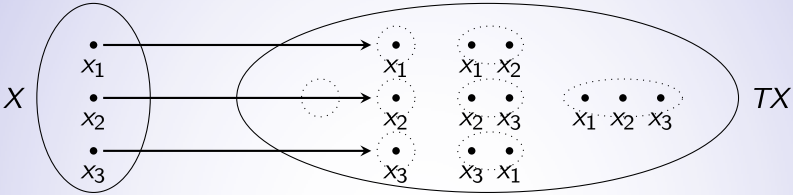
Monads as extensions



Monads as extensions



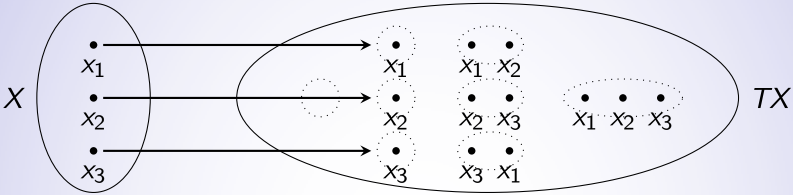
Monads as extensions



A natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow T$ consists of:

1. To each X a map $\eta_X : X \to TX$, usually monic.

Monads as extensions



A natural transformation $\eta : \mathrm{id}_C \Rightarrow T$ consists of:

1. To each X a map $\eta_X : X \to TX$, usually monic.
2. This diagram must commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ TX & \xrightarrow{Tf} & TY \end{array}$$

Monads as extensions

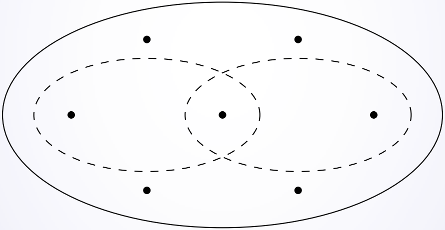
A natural transformation $\mu : TT \Rightarrow T$, is:

1. For each X a map $\mu_X : TTX \to TX$;
2. Again a naturality diagram as before.

Monads as extensions

A natural transformation $\mu : TT \Rightarrow T$, is:

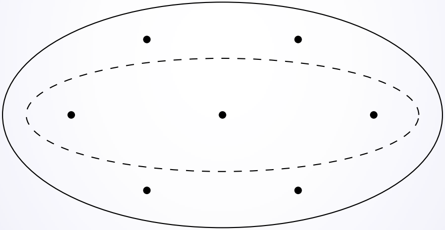
1. For each X a map $\mu_X : TTX \to TX$;
2. Again a naturality diagram as before.



Monads as extensions

A natural transformation $\mu : TT \Rightarrow T$, is:

1. For each X a map $\mu_X : TTX \to TX$;
2. Again a naturality diagram as before.



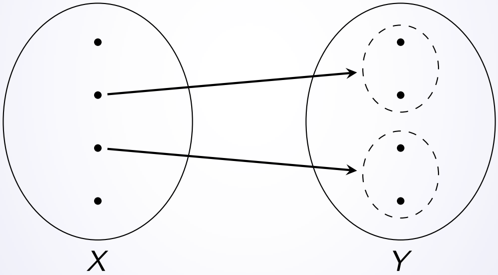
Monads as extensions

Definition:
Let T be a monad on C. A *Kleisli morphism* from X to Y is a morphism $X \to TY$.

Monads as extensions

Definition:

Let T be a monad on C. A *Kleisli morphism* from X to Y is a morphism $X \to TY$.



Monads as extensions

Definition:

Given Kleisli morphisms $k : X \to TY$ and $h : Y \to TZ$, their *Kleisli composition* is the morphism $h \circ_{kl} k$ given by:

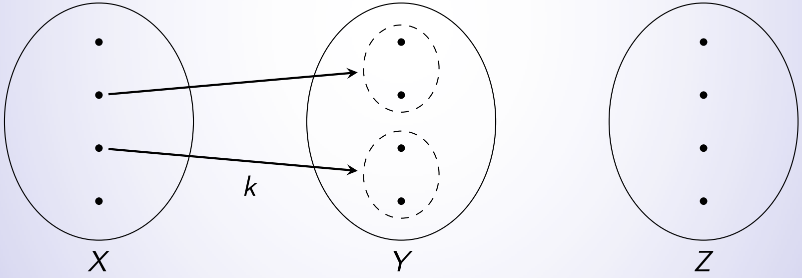
$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$

Monads as extensions

Definition:

Given Kleisli morphisms $k : X \to TY$ and $h : Y \to TZ$, their *Kleisli composition* is the morphism $h \circ_{kl} k$ given by:

$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$

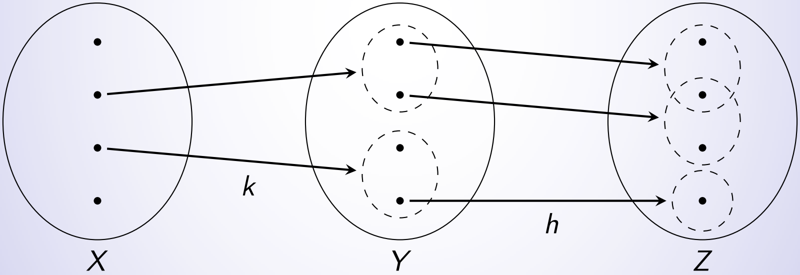


Monads as extensions

Definition:

Given Kleisli morphisms $k : X \to TY$ and $h : Y \to TZ$, their *Kleisli composition* is the morphism $h \circ_{kl} k$ given by:

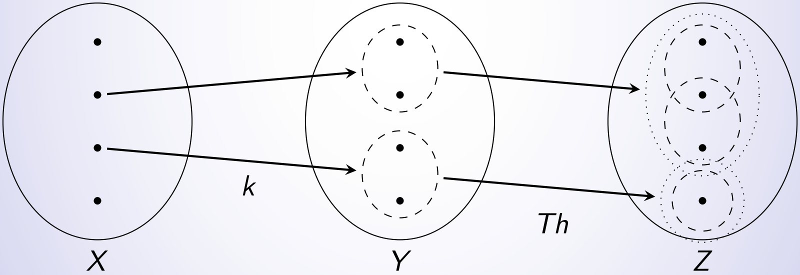
$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$



Monads as extensions

Definition:
Given Kleisli morphisms $k : X \to TY$ and $h : Y \to TZ$, their *Kleisli composition* is the morphism $h \circ_{kl} k$ given by:

$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$

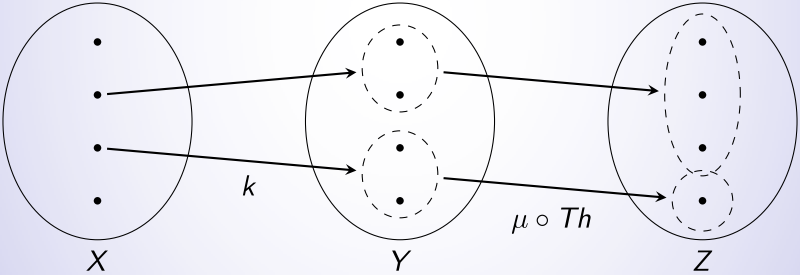


Monads as extensions

Definition:

Given Kleisli morphisms $k : X \to TY$ and $h : Y \to TZ$, their *Kleisli composition* is the morphism $h \circ_{kl} k$ given by:

$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$

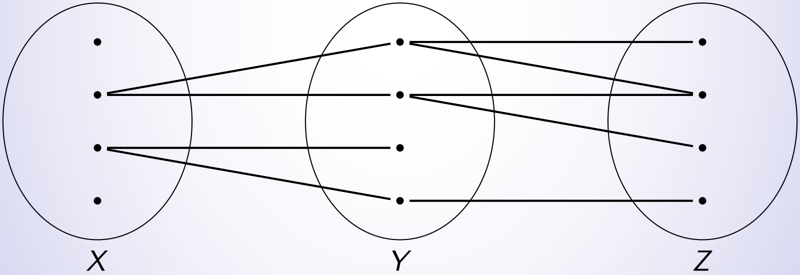


Monads as extensions

Definition:

Given Kleisli morphisms $k : X \to TY$ and $h : Y \to TZ$, their *Kleisli composition* is the morphism $h \circ_{kl} k$ given by:

$$X \xrightarrow{k} TY \xrightarrow{Th} TTZ \xrightarrow{\mu} TZ$$



Monads as extensions

Exercise:

Prove that Kleisli morphisms form a category thanks to the commutativity of these diagrams:

$$
\begin{array}{ccc}
TX & \xrightarrow{T\eta} & TTX \\
& \searrow^{=} & \downarrow{\mu} \\
& & TX
\end{array}
\qquad
\begin{array}{ccc}
TX & \xrightarrow{\eta T} & TTX \\
& \searrow^{=} & \downarrow{\mu} \\
& & TX
\end{array}
\qquad
\begin{array}{ccc}
TTTX & \xrightarrow{T\mu} & TTX \\
\downarrow{\mu T} & & \downarrow{\mu} \\
TTX & \xrightarrow{\mu} & TX
\end{array}
$$

where the identity morphisms of the Kleisli category are given by the units $\eta : X \to TX$.

Probability monads

Idea [Giry, 1982]:

Spaces of "random elements" generalizing usual elements.

Probability monads

Idea [Giry, 1982]:

Spaces of “random elements” generalizing usual elements.

- Base category C

Probability monads

Idea [Giry, 1982]:

Spaces of "random elements" generalizing usual elements.

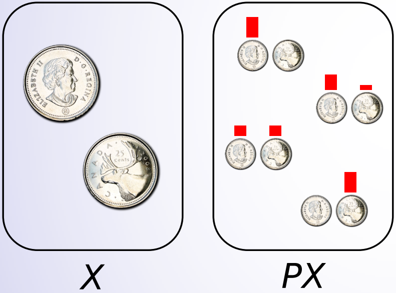
X

- Base category C

Probability monads

Idea [Giry, 1982]:

Spaces of "random elements" generalizing usual elements.

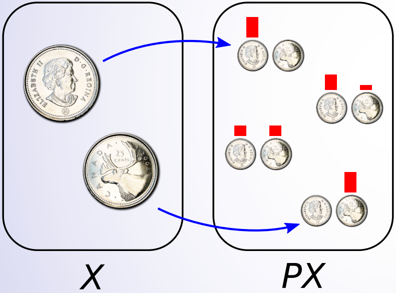


- Base category C
- Functor $X \mapsto PX$

Probability monads

Idea [Giry, 1982]:

Spaces of "random elements" generalizing usual elements.



- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$

Probability monads

Idea [Giry, 1982]:

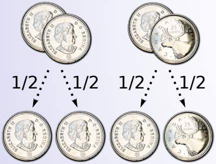
Spaces of "random elements" generalizing usual elements.

- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$

Probability monads

Idea [Giry, 1982]:

Spaces of "random elements" generalizing usual elements.

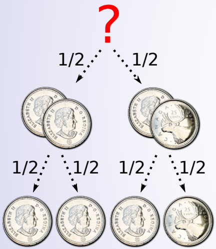


- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$

Probability monads

Idea [Giry, 1982]:

Spaces of “random elements” generalizing usual elements.

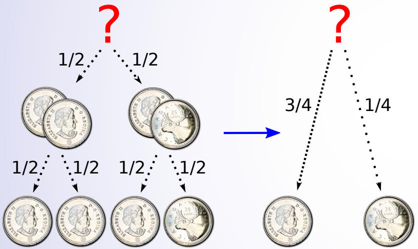


- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$

Probability monads

Idea [Giry, 1982]:

Spaces of "random elements" generalizing usual elements.

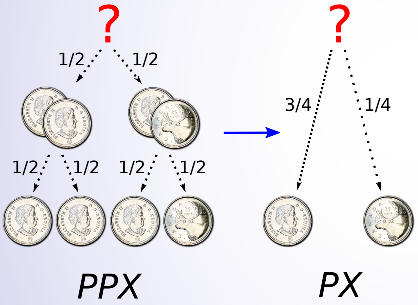


- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$

Probability monads

Idea [Giry, 1982]:

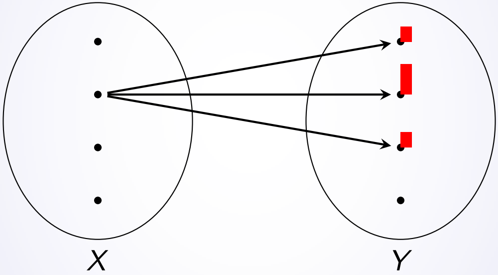
Spaces of "random elements" generalizing usual elements.



- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$
- Composition $E : PPX \to PX$

Probability monads

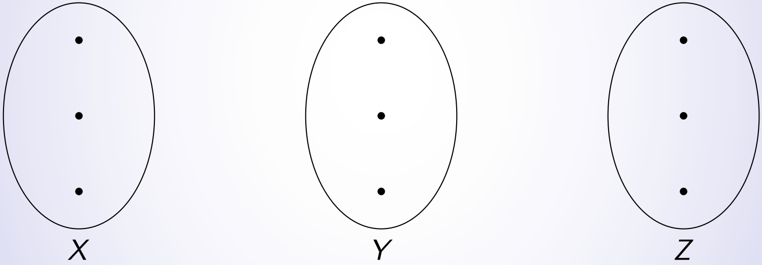
A Kleisli morphism from X to Y is a morphism $X \to PY$. We can interpret this as a "random function" or "random transition".



Probability monads

Given Kleisli morphisms $k : X \to PY$ and $h : Y \to PZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

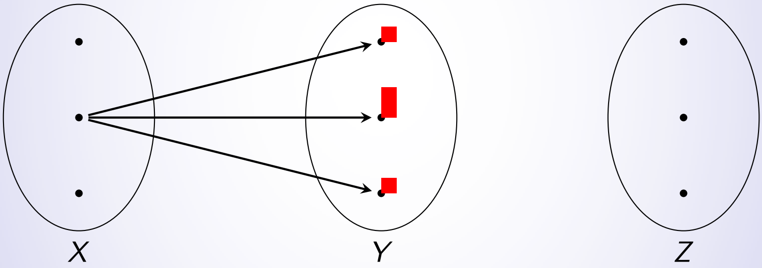
$$X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ$$



Probability monads

Given Kleisli morphisms $k : X \to PY$ and $h : Y \to PZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

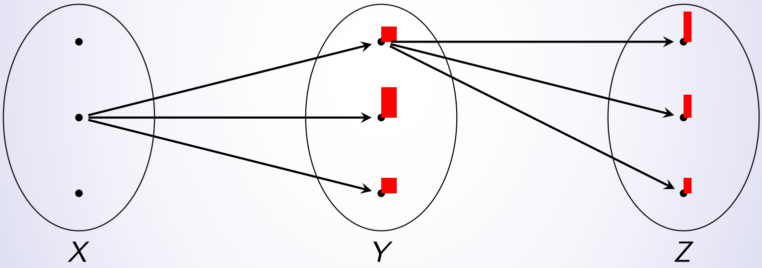
$$X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ$$



Probability monads

Given Kleisli morphisms $k : X \to PY$ and $h : Y \to PZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

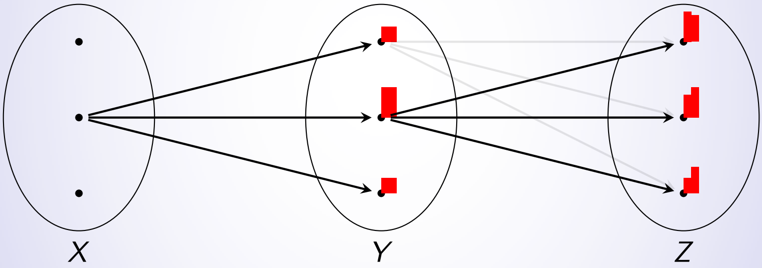
$$X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ$$



Probability monads

Given Kleisli morphisms $k : X \to PY$ and $h : Y \to PZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

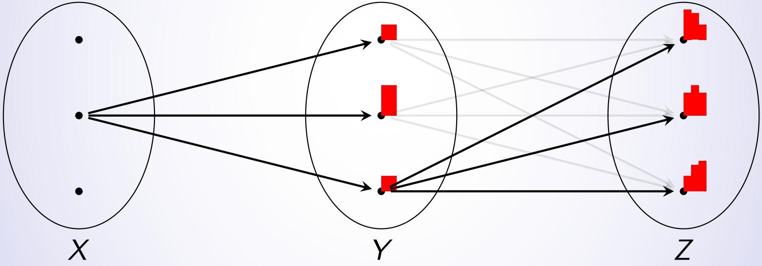
$$X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ$$



Probability monads

Given Kleisli morphisms $k : X \to PY$ and $h : Y \to PZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

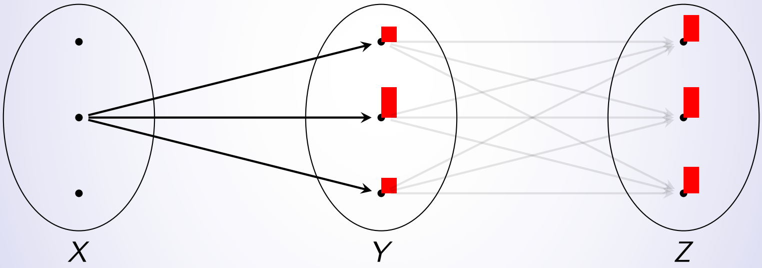
$$X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ$$



Probability monads

Given Kleisli morphisms $k : X \to PY$ and $h : Y \to PZ$, their Kleisli composition is the morphism $h \circ_{kl} k$ given by:

$$X \xrightarrow{k} PY \xrightarrow{Ph} PPZ \xrightarrow{E} PZ$$



The distribution monad on Set

Definition:

Let X be a set. A *f.s. distribution* on X is a function $p : X \to [0,1]$ such that

- It is nonzero for finitely many $x \in X$;
- $\sum_{x \in X} p(x) = 1$.

We denote by DX the set of f.s. distributions on X.

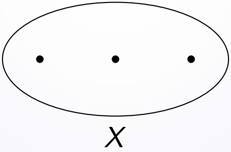
The distribution monad on Set

Definition:

Let X be a set. A *f.s. distribution* on X is a function $p : X \to [0, 1]$ such that

- It is nonzero for finitely many $x \in X$;
- $\sum_{x \in X} p(x) = 1$.

We denote by DX the set of f.s. distributions on X.



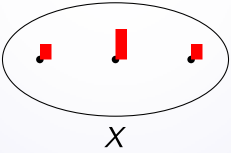
The distribution monad on Set

Definition:

Let X be a set. A *f.s. distribution* on X is a function $p : X \to [0, 1]$ such that

- It is nonzero for finitely many $x \in X$;
- $\sum_{x \in X} p(x) = 1$.

We denote by DX the set of f.s. distributions on X.



The distribution monad on Set

Definition:

Let $f : X \to Y$ be a function and $p \in DX$. The *pushforward of p along f* is the distribution $f_* p \in DY$ given by

$$f_* p(y) := \sum_{x \in f^{-1}(y)} p(x).$$

We denote the map $f_* : DX \to DY$ by Df, this makes D a functor.

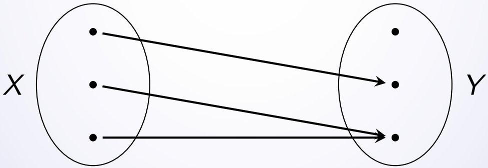
The distribution monad on Set

Definition:

Let $f : X \to Y$ be a function and $p \in DX$. The *pushforward of* p *along* f is the distribution $f_* p \in DY$ given by

$$f_* p(y) := \sum_{x \in f^{-1}(y)} p(x).$$

We denote the map $f_* : DX \to DY$ by Df, this makes D a functor.



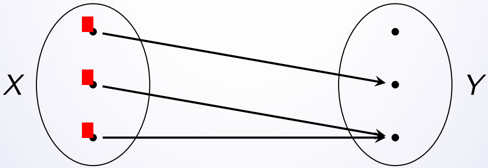
The distribution monad on Set

Definition:

Let $f : X \to Y$ be a function and $p \in DX$. The *pushforward of p along f* is the distribution $f_* p \in DY$ given by

$$f_* p(y) := \sum_{x \in f^{-1}(y)} p(x).$$

We denote the map $f_* : DX \to DY$ by Df, this makes D a functor.



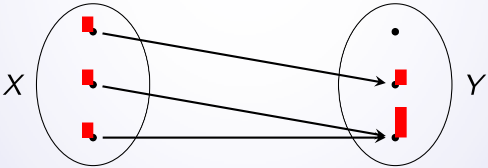
The distribution monad on Set

Definition:

Let $f : X \to Y$ be a function and $p \in DX$. The *pushforward of p along f* is the distribution $f_* p \in DY$ given by

$$f_* p(y) := \sum_{x \in f^{-1}(y)} p(x).$$

We denote the map $f_* : DX \to DY$ by Df, this makes D a functor.



The distribution monad on Set

Definition:

Let X be a set. The map $\delta : X \to DX$ maps $x \in X$ to the distribution $\delta_x \in DX$ given by

$$\delta_x(y) = \begin{cases} 1 & y = x; \\ 0 & y \neq x. \end{cases}$$

This gives a natural map $\delta : X \to DX$, a component of the unit of the monad.

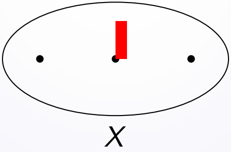
The distribution monad on Set

Definition:

Let X be a set. The map $\delta : X \to DX$ maps $x \in X$ to the distribution $\delta_x \in DX$ given by

$$\delta_x(y) = \begin{cases} 1 & y = x; \\ 0 & y \neq x. \end{cases}$$

This gives a natural map $\delta : X \to DX$, a component of the unit of the monad.



The distribution monad on Set

Definition:

Let X be a set. Given $\xi \in DDX$, define $E\xi \in DX$ to be distribution given by

$$E\xi(x) := \sum_{p \in DX} p(x)\, \xi(p).$$

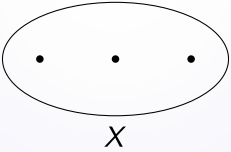
This gives a natural map $E : DDX \to DX$, a component of the multiplication of the monad.

The distribution monad on Set

Definition:
Let X be a set. Given $\xi \in DDX$, define $E\xi \in DX$ to be distribution given by

$$E\xi(x) := \sum_{p \in DX} p(x)\, \xi(p).$$

This gives a natural map $E : DDX \to DX$, a component of the multiplication of the monad.



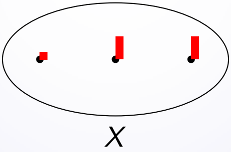
The distribution monad on Set

Definition:

Let X be a set. Given $\xi \in DDX$, define $E\xi \in DX$ to be distribution given by

$$E\xi(x) := \sum_{p \in DX} p(x)\,\xi(p).$$

This gives a natural map $E : DDX \to DX$, a component of the multiplication of the monad.



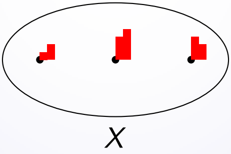
The distribution monad on Set

Definition:

Let X be a set. Given $\xi \in DDX$, define $E\xi \in DX$ to be distribution given by

$$E\xi(x) := \sum_{p \in DX} p(x)\, \xi(p).$$

This gives a natural map $E : DDX \to DX$, a component of the multiplication of the monad.

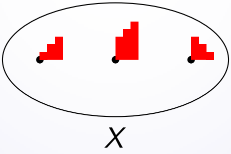


The distribution monad on Set

Definition:
Let X be a set. Given $\xi \in DDX$, define $E\xi \in DX$ to be distribution given by

$$E\xi(x) := \sum_{p \in DX} p(x)\,\xi(p).$$

This gives a natural map $E : DDX \to DX$, a component of the multiplication of the monad.

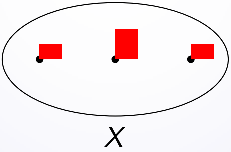


The distribution monad on Set

Definition:
Let X be a set. Given $\xi \in DDX$, define $E\xi \in DX$ to be distribution given by

$$E\xi(x) := \sum_{p \in DX} p(x)\,\xi(p).$$

This gives a natural map $E : DDX \to DX$, a component of the multiplication of the monad.



The distribution monad on Set

Kleisli morphisms:

A Kleisli morphism for D is a function $k : X \to DY$. In other words, it is function $\bar{k} : X \times Y \to [0,1]$ such that

- For each $x \in X$, $\bar{k}(x, -) : Y \to [0,1]$ is nonzero in finitely many entries;
- For each $x \in X$, $\sum_{y \in Y} \bar{k}(x, y) = 1$.

The distribution monad on Set

Kleisli morphisms:

A Kleisli morphism for D is a function $k : X \to DY$. In other words, it is function $\bar{k} : X \times Y \to [0, 1]$ such that

- For each $x \in X$, $\bar{k}(x, -) : Y \to [0, 1]$ is nonzero in finitely many entries;
- For each $x \in X$, $\sum_{y \in Y} \bar{k}(x, y) = 1$.

Kleisli composition:

The Kleisli composition of $k : X \to DY$ and $h : Y \to DZ$ is given by the Chapman-Kolmogorov equation:

$$(h \circ_{kl} k)(x, z) = \sum_{y \in Y} k(x, y)\, h(y, z).$$

The Giry monad on Meas

Let X be a measurable space. Define PX to be

- The set of probability measures on X

The Giry monad on Meas

Let X be a measurable space. Define PX to be

- The set of probability measures on X
- Equipped with the σ-algebra generated by the evaluation functions $\varepsilon_A : PX \to \mathbb{R}$ given by

$$p \longmapsto p(A)$$

  for all $A \subseteq X$ measurable.

The Giry monad on Meas

Let X be a measurable space. Define PX to be

- The set of probability measures on X
- Equipped with the σ-algebra generated by the evaluation functions $\varepsilon_A : PX \to \mathbb{R}$ given by

$$p \longmapsto p(A)$$

  for all $A \subseteq X$ measurable.
- Equivalently, the σ-algebra is generated by the "integration" functions $\varepsilon_f : PX \to \mathbb{R}$ given by

$$p \longmapsto \int f \, dp,$$

  for all $f : X \to [0,1]$ measurable.

The Giry monad on Meas

Functoriality:

Let $f : X \to Y$ be a measurable function. Given a measure $p \in PX$, recall that the pushforward measure $f_* p \in PY$ is given by

$$f_* p(B) := p(f^{-1}(B)).$$

We get a measurable map $Pf : PX \to PY$ which makes P a functor.

The Giry monad on Meas

Functoriality:

Let $f : X \to Y$ be a measurable function. Given a measure $p \in PX$, recall that the pushforward measure $f_* p \in PY$ is given by

$$f_* p(B) := p(f^{-1}(B)).$$

We get a measurable map $Pf : PX \to PY$ which makes P a functor.

Unit:

Given a measurable space X, to each $x \in X$ we can give the Dirac delta measure $\delta_x \in PX$. This gives a measurable map $\delta : X \to PX$, which is natural, and forms a component of the unit of the monad.

The Giry monad on Meas

Multiplication:

Given a measurable space X and a measure $\pi \in PPX$, we define the measure $E\pi \in PX$ by

$$E\pi(A) := \int_{PX} p(A) d\pi(p),$$

This gives a measurable map $E : PPX \to PX$ which is natural in X and forms a component of the monad multiplication.

The Giry monad on Meas

Kleisli morphisms:

A Kleisli morphism is a measurable map $k : X \to PY$, in other words, a *Markov kernel* between X and Y. Denote $k(x) \in PY$ by k_x.

The Giry monad on Meas

Kleisli morphisms:

A Kleisli morphism is a measurable map $k : X \to PY$, in other words, a *Markov kernel* between X and Y. Denote $k(x) \in PY$ by k_x.

Kleisli composition:

The composition of Kleisli morphisms reproduces the Chapman-Kolmogorov equation for general measures. Given $k : X \to PY$ and $h : Y \to PZ$, we get that

$$(h \circ_{kl} k)(x)(C) = \int_Y h_y(C) dk_x(y)$$

for each $x \in X$ and for each $C \subseteq Z$ measurable.

Other probability monads

Category	Monad (P)	Points of PX
Set	Distribution monad	f.s. distributions
Meas	Giry monad	probability measures
Pol	Giry monad	Borel probability measures
QBS	Prob. monad	Eq. classes of R.V.s
DCPO	Prob. powerdomain	cont. valuations
Top	Ext. prob. PD	cont. valuations
Top	Prob. monad	τ-smooth Borel prob. measures
CHaus	Radon monad	Radon prob. measures
Met	Kantorovich monad	Radon prob. measures of FFM

More on the nLab, “probability monad” [nLab article].

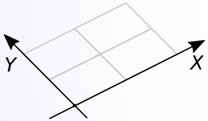
Joints and marginals

Idea:
Probability theory is mostly about *interactions* of random variables.

- Composite states
  $X \times Y$

Joints and marginals

Idea:
Probability theory is mostly about *interactions* of random variables.

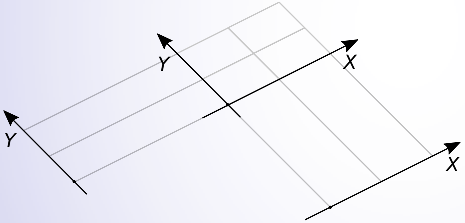


- Composite states
  $X \times Y$

Joints and marginals

Idea:

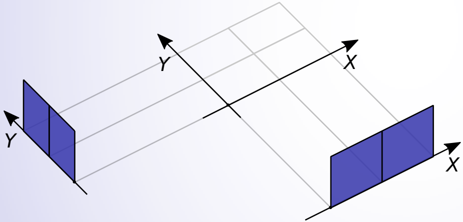
Probability theory is mostly about *interactions* of random variables.



- Composite states $X \times Y$

Joints and marginals

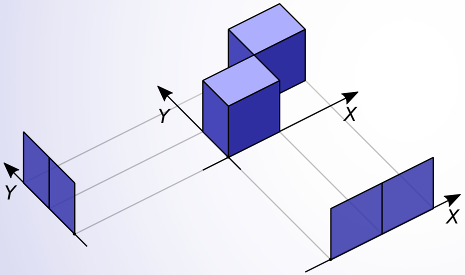
Idea:
Probability theory is mostly about *interactions* of random variables.



- Composite states $X \times Y$
- Given marginals

Joints and marginals

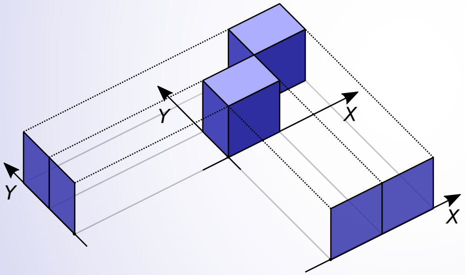
Idea:
Probability theory is mostly about *interactions* of random variables.



- Composite states $X \times Y$
- Given marginals
- Many possible joints

Joints and marginals

Idea:
Probability theory is mostly about *interactions* of random variables.

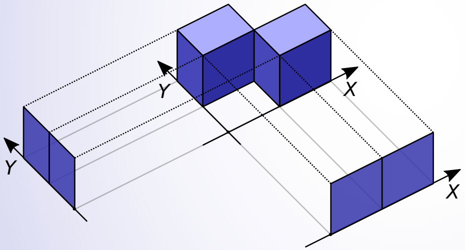


- Composite states $X \times Y$
- Given marginals
- Many possible joints

Joints and marginals

Idea:

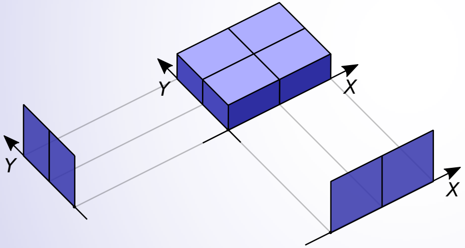
Probability theory is mostly about *interactions* of random variables.



- Composite states $X \times Y$
- Given marginals
- Many possible joints

Joints and marginals

Idea:
Probability theory is mostly about *interactions* of random variables.



- Composite states $X \times Y$
- Given marginals
- Many possible joints
- One canonical choice of "independence"

Joints and marginals

Idea:

Given objects X and Y, a probability distribution on $X \times Y$ is not just pair of distributions on X and Y separately.
However, given $p \in PX$ and $q \in PY$, we get a measure $p \otimes q \in P(X \times Y)$.

$$PX \times PY \xrightarrow{\nabla} P(X \times Y)$$

Joints and marginals

Idea:

Given objects X and Y, a probability distribution on $X \times Y$ is not just pair of distributions on X and Y separately.
However, given $p \in PX$ and $q \in PY$, we get a measure $p \otimes q \in P(X \times Y)$.

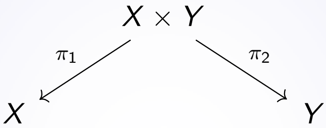
$$PX \times PY \xrightarrow{\nabla} P(X \times Y)$$

This gives a *monoidal structure* to the probability monad.
(Technically, we need ∇ together with a map $1 \to P1$, but for probability monads 1 and $P1$ are uniquely isomorphic.)

Joints and marginals

$$
\begin{array}{ccc}
PX \times PY \times PZ & \xrightarrow{\nabla \times \mathrm{id}} & P(X \times Y) \times PZ \\
\Big\downarrow{\scriptstyle \mathrm{id} \times \nabla} & & \Big\downarrow{\scriptstyle \nabla} \\
PX \times P(Y \times Z) & \xrightarrow{\nabla} & P(X \times Y \times Z)
\end{array}
$$

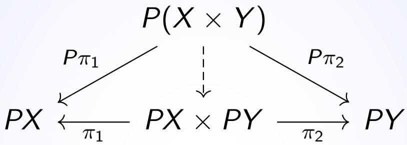
Joints and marginals



Joints and marginals

$$PX \xleftarrow{P\pi_1} P(X \times Y) \xrightarrow{P\pi_2} PY$$

Joints and marginals



Joints and marginals

$$\begin{array}{ccccc} & & P(X \times Y) & & \\ & \swarrow^{P\pi_1} & \downarrow{\scriptstyle \Delta} & \searrow^{P\pi_2} & \\ PX & \xleftarrow[\pi_1]{} & PX \times PY & \xrightarrow[\pi_2]{} & PY \end{array}$$

$$\begin{CD} P(X \times Y \times Z) @>{\Delta \times \mathrm{id}}>> PX \times P(Y \times Z) \\ @V{\mathrm{id} \times \Delta}VV @VV{\Delta}V \\ P(X \times Y) \times PZ @>{\Delta}>> PX \times PY \times PZ \end{CD}$$

Joints and marginals

Operations on distributions:

Let $f : X \times Y \to Z$ be a binary function. Then we can form the map

$$PX \times PY \xrightarrow{\nabla} P(X \times Y) \xrightarrow{Pf} PZ$$

Joints and marginals

Operations on distributions:

Let $f : X \times Y \to Z$ be a binary function. Then we can form the map

$$PX \times PY \xrightarrow{\nabla} P(X \times Y) \xrightarrow{Pf} PZ$$

For example, the addition as a map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ gives the *convolution* of real-valued random variables.

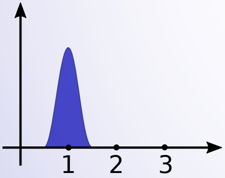
Joints and marginals

Operations on distributions:

Let $f : X \times Y \to Z$ be a binary function. Then we can form the map

$$PX \times PY \xrightarrow{\nabla} P(X \times Y) \xrightarrow{Pf} PZ$$

For example, the addition as a map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ gives the *convolution* of real-valued random variables.



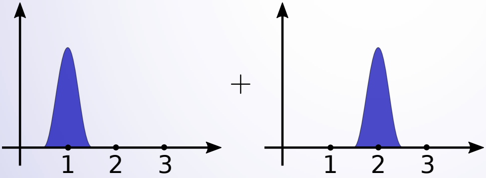
Joints and marginals

Operations on distributions:

Let $f : X \times Y \to Z$ be a binary function. Then we can form the map

$$PX \times PY \xrightarrow{\nabla} P(X \times Y) \xrightarrow{Pf} PZ$$

For example, the addition as a map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ gives the *convolution* of real-valued random variables.



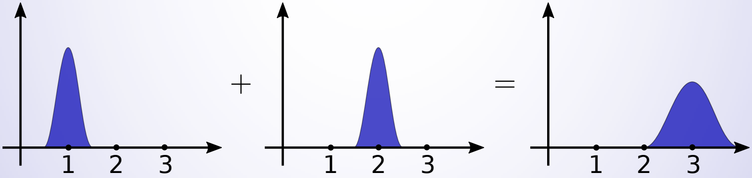
Joints and marginals

Operations on distributions:

Let $f : X \times Y \to Z$ be a binary function. Then we can form the map

$$PX \times PY \xrightarrow{\nabla} P(X \times Y) \xrightarrow{Pf} PZ$$

For example, the addition as a map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ gives the *convolution* of real-valued random variables.



Some references

van Breugel, F. (2005).
The Metric Monad for Probabilistic Nondeterminism.
www.cse.yorku.ca/~franck/research/drafts/monad.pdf.

Fritz, T. and Perrone, P. (2018).
Bimonoidal structure of probability monads.
Proceedings of MFPS 34.

Fritz, T. and Perrone, P. (2020).
Monads, partial evaluations, and rewriting.
Proceedings of MFPS 36.

Giry, M. (1982).
A Categorical Approach to Probability Theory.
In *Categorical aspects of topology and analysis*, volume 915 of *Lecture Notes in Mathematics*.

Heunen, C., Kammar, O., Staton, S., and Yang, H. (2017).
A convenient category for higher-order probability theory.
Proceedings of LICS'17, (77):1–12.

Jacobs, B. (2018).
From probability monads to commutative effectus.
Journal of Logical and Algebraic Methods in Programming, 94:200–237.

Keimel, K. (2008).
The monad of probability measures over compact ordered spaces and its Eilenberg-Moore algebras.
Topology and its Applications, 156(2):227–239.

nLab article.
Monads of probability, measures and valuations.
ncatlab.org/nlab/show/probability+monad.

Perrone, P. (2018).
Categorical Probability and Stochastic Dominance in Metric Spaces.
PhD thesis, University of Leipzig.
www.paoloperrone.org/phdthesis.pdf.

Perrone, P. (2019).
Notes on category theory with examples from basic mathematics.
arXiv:1912.10642.