Resource theories in the asymptotic and catalytic regime

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Asymptotics in resource theories

- In resource theories, asymptotics are usually easy.
- It makes sense to study asymptotics first.
- Historical examples:
  - Monge 1781: **optimal transportation**.
  - Carnot 1824: **Carnot efficiency**.
  - Kantorovich 1939: **linear programming**.
  - Shannon 1948: **channel coding theorem**.
- For example, Shannon's theorem is asymptotic and has resulted in the development of coding theory.

Majorization

Let

$$p = (p_1 \geq \ldots \geq p_n > 0), \qquad q = (q_1 \geq \ldots \geq q_m > 0)$$

be probability vectors.

Proposition

The following are equivalent:

- ▷ There is a bistochastic matrix T such that

$$Tp = q.$$

- ▷ For all k,

$$\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i.$$

When these hold, we say that p **majorizes** q,

$$p \succeq q.$$

Majorization

- Intuition: q contains more randomness than p.

Theorem (Nielsen '99)

For bipartite pure-state entanglement,

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$$

if and only if the spectra of the reduced density matrices display majorization,

$$|\psi\rangle\langle\psi|_A^{\downarrow} \succeq |\phi\rangle\langle\phi|_A^{\downarrow}.$$

- Provides an easy-to-use criterion for LOCC.
- What about the asymptotics?
- Since $|\psi\rangle \mapsto |\psi\rangle\langle\psi|_A^{\downarrow}$ preserves tensor products, it's enough to consider asymptotic majorizarion.

The **Rényi entropies**

$$H_\alpha(p) := \frac{1}{1-\alpha} \log \left(\sum_i p_i^\alpha \right)$$

will play an important role.

Special cases:

$$H_0(p) = \log |\{i \mid p_i > 0\}|$$

$$H_1(p) = -\sum_i p_i \log p_i$$

$$H_\infty(p) = -\log \max_i p_i.$$

Asymptotic majorization

Theorem (Jensen '19)

If

$$H_\alpha(p) > H_\alpha(q) \quad \forall \alpha \in [0, \infty],$$

then

$$p^{\otimes n} \preceq q^{\otimes n} \quad \forall n \gg 1.$$

Conversely, $p^{\otimes n} \preceq q^{\otimes n}$ for some n implies $H_\alpha(p) \geq H_\alpha(q)$.

- Implies a rate formula (TF '17):

$$R(|\psi\rangle \to |\phi\rangle) = \inf_\alpha \frac{H_\alpha(|\psi\rangle\langle\psi|_A^\downarrow)}{H_\alpha(|\phi\rangle\langle\phi|_A^\downarrow)}$$

Catalytic majorization

Theorem (Klimesh '07, Turgut '07)

If

$$H_\alpha(p) > H_\alpha(q) \quad \forall \alpha \in [0, \infty],$$

then

$$\exists r \; : \; p \otimes r \preceq q \otimes r.$$

Conversely, $p \otimes r \preceq q \otimes r$ implies $H_\alpha(p) \geq H_\alpha(q)$.

- Why are asymptotic and catalytic majorization essentially equivalent?
- And what's special about the H_α?
- I will answer these questions and explain the general theorem.

- The Rényi entropies are **additive monotones**:
  - Additivity:
$$H_\alpha(p \otimes q) = H_\alpha(p) + H_\alpha(q).$$
  - Monotonicity:
$$p \preceq q \quad \Longrightarrow \quad H_\alpha(p) \geq H_\alpha(q).$$
- There are other additive monotones, such as $H_0 + H_1$.
- So something is still missing.

- Instead of the H_α, consider just

$$\|p\|_\alpha = \sum_\alpha p_i^\alpha$$

for $\alpha \neq 1, \infty$.

- We can use $\|p\|_\alpha > \|q\|_\alpha$ instead of $H_\alpha(p) > H_\alpha(q)$.

- The $\|\cdot\|_\alpha$ satisfy multiplicativity

$$\|p \otimes q\|_\alpha = \|p\|_\alpha \|q\|_\alpha$$

and monotonicity,

$$p \preceq q \quad \Longrightarrow \quad \|p\|_\alpha \geq \|q\|_\alpha.$$

- They also satisfy **additivity under direct sum**

$$\|p \oplus q\|_\alpha = \|p\|_\alpha + \|q\|_\alpha$$

  if we allow unnormalized probability vectors.

- Therefore they are **monotone semiring homomorphisms**

$$\mathsf{Major} \longrightarrow \mathbb{R}_+.$$

Definition

An **ordered semiring** $(S, +, \cdot, \geq)$ is an algebraic structure satisfying the usual equations, and

$$x \geq y \quad \Longrightarrow \quad x + z \geq y + z, \quad xz \geq yz,$$

- Major is ordered semiring of probability vectors with $(\oplus, \otimes, \preceq)$.

- But what about H_1 and H_∞?
- For H_∞, consider instead

$$\|p\|_\infty = \max_i p_i.$$

- Still a multiplicative monotone.
- But now

$$\|p \oplus q\|_\infty = \max(\|p\|_\infty, \|q\|_\infty).$$

Definition

The **tropical reals** are the ordered semiring

$$\mathbb{TR}_+ := ([0,\infty), \max, \cdot, \geq).$$

- What about H_1?
- No well-behaved "exponential" exists.
- Still have additivity

$$H_1(p \oplus q) = H_1(p) + H_1(q)$$

  and monotonicity.
- **Not** the usual "additivity" of entropy!
- On tensor products, we have the **Leibniz rule**

$$H_1(p \otimes q) = H_1(p)\,\|q\|_1 + \|p\|_1 H_1(q).$$

  We say that H_1 is a **derivation** at $\|\cdot\|_1$.

To summarize, the following types of monotones are important:

- Semiring homomorphisms Major $\to \mathbb{R}_+$.
- Semiring homomorphisms Major $\to \mathbb{TR}_+$.
- The derivations at $\|\cdot\|_1$.

Theorem (With Farooq, Haapasalo, Tomamichel)

(a) The $\|\cdot\|_\alpha$ for $\alpha \neq \infty$ are all the monotone homs Major $\to \mathbb{R}_+$.

(b) $\|\cdot\|_\infty$ is the only monotone hom Major $\to \mathbb{T}\mathbb{R}_+$.

(c) H_1 is (essentially) the only derivation at $\|\cdot\|_1$.

- Sketch: Let f : Major $\to \mathbb{R}_+$ be a monotone hom. Then

  $$f(p_1, \ldots, p_n) = f(p_1) + \cdots + f(p_n).$$

  So it's enough to look at vectors of length 1!

- On those, we have the **Cauchy functional equation**

  $$f(pq) = f(p)f(q),$$

  whose only well-behaved solutions are the power functions $f(p) = p^\alpha$.

General case

- Now let's generalize to a statement that should apply to many other resource theories too!
- So let S be any suitably well-behaved ordered semiring and

$$\|\cdot\| \,:\, S \longrightarrow \mathbb{R}_+$$

  any homomorphism such that

$$p \leq q \quad \Longrightarrow \quad \|p\| = \|q\|.$$

- Let us say that the **relevant monotones** are those described above:
  - Monotone homs $S \to \mathbb{R}_+$.
  - Monotone homs $S \to \mathbb{TR}_+$.
  - Monotone derivations $S \to \mathbb{R}$ at $\|\cdot\|$.

Theorem (Vergleichsstellensatz)

Let nonzero $x, y \in S$ satisfy $\|x\| = \|y\|$. If

$$f(x) > f(y)$$

for all relevant monotones f, then:

- There is nonzero c such that

$$xc \geq yc.$$

- If x is sufficiently generic,

$$x^n \geq y^n \qquad \forall n \gg 1.$$

Conversely, if $xc \geq yc$ for nonzero c or $x^x \geq y^n$ for some $n \geq 1$, then

$$f(x) \geq f(y)$$

for all relevant monotones.

- Provides an almost tight criterion for asymptotic and catalytic convertibility very generally.
- In particular, shows that asymptotic and catalytic convertibility are essentially equivalent.
- Recovers known statements on asymptotic and catalytic majorization.
- **Other applications?**

Application to representation theory

- Representations form an ordered semiring with respect to $\oplus$, $\otimes$ and containment.

Theorem

For representations of $SU(2)$, the relevant monotones are parametrized by $\alpha \in [0, \infty]$,

$$f_\alpha(U) := \sum_i \frac{\sinh(\alpha \dim(U_i))}{\sinh \alpha}.$$

where $U = \bigoplus_\beta U_i$ with irreducible U_i,

Application to representation theory

- How many spin-$\frac{1}{2}$ qubits do we need in order to simulate $n \gg 1$ systems of spin 1, respecting the symmetry?

- This can now be computed explicitly: we need Rn, where the **rate** R is

$$R = R(\mathrm{spin}\,\tfrac{1}{2} \to \mathrm{spin}\,1) = \inf_{\alpha \geq 0} \frac{\log \frac{\sinh(2\alpha)}{\sinh(a)}}{\log \frac{\sinh(3\alpha)}{\sinh(\alpha)}} = \frac{1}{2}.$$

Final thoughts 1

- Ordered semirings provide a general framework for resource theories with powerful mathematical results.

- Applying these gives non-constructive proofs for the existence of information processing protocols.

- No notion of "free resource" or "free operation" is relevant!

Final thoughts 2

- **Epsilonification** is the big open problem: build in approximations in the asymptotics such that e.g. Shannon's channel coding theorem comes out of a general theory as well.

Conjecture

In an epsilonified resource theory, **only the derivations** are relevant.

- E.g. in thermodynamics, this should amount to the free energies, since both entropy and energy are derivations:

$$E(\rho \otimes \eta) = E(\rho)\,\|\eta\|_1 + \|\rho\|_1 E(\eta).$$

- Unfortunately, finding the "right" definitions has turned out to be very difficult.

Quantum thermodynamics[3]

Consider a system with finite-dimensional Hilbert space $\mathbb{C}^d$ and Hamiltonian H. A state ρ has energy $E(\rho) = \mathrm{tr}(\rho H)$ and entropy $S(\rho) = -\mathrm{tr}(\rho \log \rho)$.

Theorem

For states ρ and σ, the following are equivalent:

(a) There exists an ancilla system of size $o(n)$ with state η and Hamiltonian H_{anc} satisfying $\|H_{\mathrm{anc}}\| \leq o(n)$ and an energy-preserving unitary U with

$$\left\| \mathrm{Tr}_{\mathrm{anc}} \left[U(\boldsymbol{\rho}^{\otimes n} \otimes \eta) U^\dagger \right] - \boldsymbol{\sigma}^{\otimes n} \right\|_1 \overset{n \to \infty}{\longrightarrow} 0.$$

(b) The states have equal energy and entropy,

$$E(\rho) = E(\sigma), \qquad S(\rho) = S(\sigma).$$

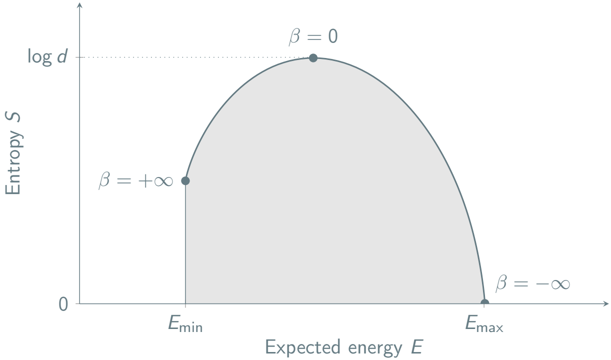
[1] Carlo Sparaciari, Jonathan Oppenheim, and Tobias Fritz. "A Resource Theory for Work and Heat". In: **Phys. Rev. A 96, 052112** (2017). arXiv:1607.01302.

Definition

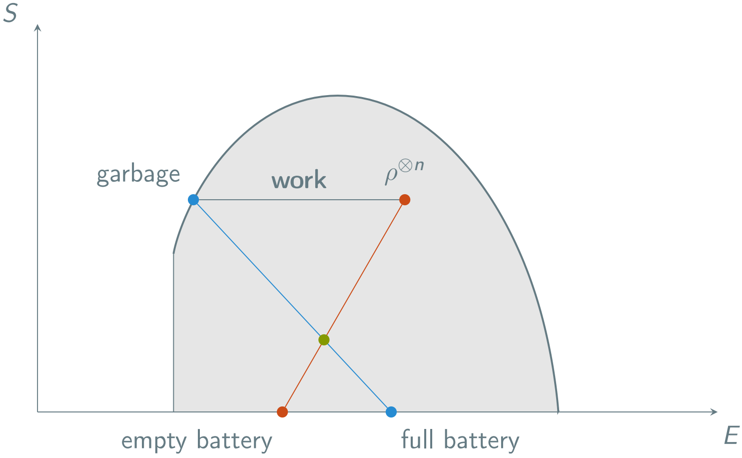
A **macrostate** is an equivalence class of states with respect to asymptotic interconvertibility as in the theorem.

By the theoerm, macrostates correspond to pairs (E, S) that can be jointly attained.

The set of macrostates forms the **energy-entropy diagram**:



Example: The maximum extractable work per copy of a state ρ is given by the horizontal distance to the boundary:



Similarly: analysis of heat engines with finite (but large) reservoirs!

Linear programming

Press release

14 October 1975

The Royal Swedish Academy of Sciences has decided to award the Prize in Economic Science in Memory of Alfred Nobel for 1975 in equal shares to

Professor **Leonid Kantorovich**, USSR,
and
Professor **Tjalling C. Koopmans**, USA,

for their contributions to the theory of optimum allocation of resources.

Optimum Allocation of Resources
Leonid Kantorovich and Tjalling Koopmans have both done their most important scientific work in the field of normative economic theory, *i.e.*, the theory of the optimum allocation of resources. As the starting point of their work in this field, both have studied the problem – fundamental to all economic activity – of how available productive resources can be used to the greatest advantage in the production of goods and services. This field embraces such questions as what goods should be produced, what methods of production should be used and how much of current production should be consumed, and how much reserved to create new resources for future production and consumption.

Linear programming

- Linear programming is appropriate and useful whenever:
  - Resources are arbitrarily divisible.
  - They come in finitely many types.
  - Finitely many basic conversions.
- In general, all of them fail!
- So what replaces linear programming?
- I will explain some results in this direction.

What are resources?

Resources can be **converted** into each other via processes, such as:

$$\text{timber} + \text{nails} \xrightarrow{\text{Carpentry}} \text{table}$$

The details vary with the context:

- Communication:

$$\text{noisy channel} \xrightarrow{\text{Channel coding}} \text{perfect channel}$$

- Thermodynamics:

$$\text{hot gas} + \text{cold gas} \xrightarrow{\text{Carnot process}} \text{gas} + \text{work}$$

- Industrial chemistry:

$$N_2 + 3\,H_2 \xrightarrow{\text{Haber process}} 2\,NH_3$$

A mathematical theory of resources

- Pattern: **convertibility** and **combinability** of resource objects.
- One investigates questions on **catalysis**, **asymptotic rates**,...

Definition ([4])

An ordered commutative monoid is a structure $(A, +, 0, \geq)$ such that:

$$x + y = y + x \qquad (x + y) + z = x + (y + z) \qquad x + 0 = x$$

$$x \geq y \geq z \quad \Rightarrow \quad x \geq z.$$

$$x \geq y \quad \Rightarrow \quad x + z \geq y + z.$$

- No notion of "free resource" or "free operation" is needed.

[4] Tobias Fritz. "Resource convertibility and ordered commutative monoids". In: **Math. Structures Comput. Sci.** 27.6 (2017), pp. 850–938.

- Conceptual insights:
  - Catalysts = tools.
  - Allowing free use of catalysts $\cong$ having banks to borrow from.
  - Considering asymptotics $\cong$ efficiency of scale and mass production.
  - Asymptotic structure of a resource theory: **two convex cones** in a vector space.
    - One for resource objects;
    - One for resource conversion: $x \geq y$ asymptotically iff $x - y$ in this cone.
- Main open problem: how to **build in approximations?**
- However, we can still exploit the conceptual insights in quantum information theory, for example in **quantum thermodynamics**!