An approach to homological algebra up to ε

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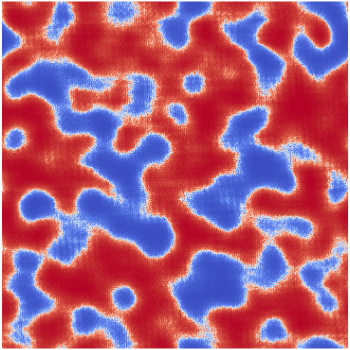


Motivation for nonabelian homological algebra

Long exact sequence of homotopy groups of a fibration:

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \cdots$$

Which category does this live in?

π_1 can be nonabelian!

$\Rightarrow$ Usual homological algebra does not apply.

Kazhdan's ε-representations (1982)

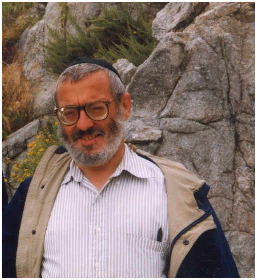
ON ε-REPRESENTATIONS

BY
D. KAZHDAN

ABSTRACT

For certain classes of groups we show that a map to the group of unitary transformations of a Hilbert space which is "almost" a homomorphism is uniformly close to a unitary representation.

V. Milman asked me the following question: Let $\rho : O(n) \to O(N)$ be a map which is "almost" a representation, that is, $|\rho(gg') - \rho(g)\rho(g')|$ is small for all $g, g' \in O(N)$. Is it true that ρ is near to an actual representation of $O(n)$? This paper is a particular answer to this question.



Definition

For G a group and E a Banach space, an ε-**representation** is a map

$$G \times E \to E$$

such that G acts by isometries, and

$$\|g(g'x) - (gg')x\| \leq \varepsilon\|x\| \qquad \forall g, g' \in G,\ x \in E.$$

Definition

Let C^n be the space of chains $G^n \to E$ with respect to the sup norm,

$$\|c\| \coloneqq \sup_{g_1,\dots,g_n \in G} \|c(g_1,\dots,g_n)\|$$

Then the usual differential $d : C^n \to C^{n+1}$ given by

$$\begin{aligned}(dc)(g_1,\dots,g_{n+1}) &\coloneqq g_1 c(g_2,\dots,g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i c(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1} c(g_1,\dots,g_n)\end{aligned}$$

satisfies

$$\left\|d^2\right\| \leq \varepsilon.$$

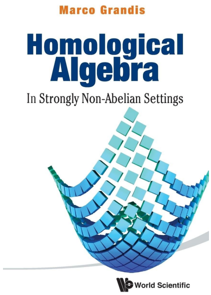
Kazhdan uses this structure to prove his main result:

THEOREM 1. *Let G be an amenable group and $\tilde{\rho}: G \to U$ be an ε-representation of G into the group U of unitary transformations of a Hilbert space H for $\varepsilon < 1/100$. Then there exists a representation $\pi: G \to U$ such that $\|\tilde{\rho}(g) - \pi(g)\| \leqq \varepsilon$ for all $g \in G$.*

He also proves that this fails for other groups, even for a finitely presented G and $\dim(H) < \infty$.

Homological algebra up to ε?

- Kazhdan's method is successful, but ad hoc.
- So what could a general theory of homological algebra "up to ε" look like?
- Grandis's framework for nonabelian homological algebra looks promising!

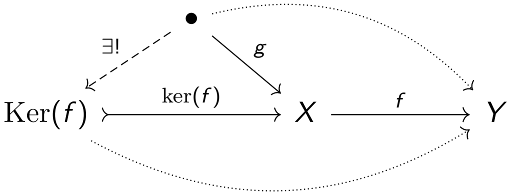


The setting

- A **category with null morphisms** is a category C together with an ideal of morphisms $\mathcal{N}$ called **null**, drawn as $\dashrightarrow$.
- A **kernel** of a morphism $f : X \to Y$ is

$$\ker(f) \;:\; \mathrm{Ker}(f) \longrightarrow X$$

such that $f \circ \ker(f)$ is null, and such that for all g,



- A **cokernel** is defined dually.

The setting

- Assume that all kernels and cokernels exist.

Enough to prove some basic things!

- Every kernel is the kernel of its cokernel.
- The collection of kernels at an object X forms the **lattice of normal subobjects**
$$\mathsf{nSub}(X),$$
with id_X as top element and $\ker(\mathrm{id}_X)$ as bottom element.

Category of groups is a good example.

- These assumptions are not enough to do homological algebra.
- Grandis requires additional axioms, resulting in the definition of **homological category**.
- In these categories, homological algebra makes sense: **snake lemma, long exact sequences, etc.**
- Examples:
  - The category of lattices and Galois connections is homological.
  - The category of groups is not homological!

Back to Kazhdan

Definition

For $\varepsilon \in (0,1)$ fixed, $\mathbf{Norm}_\varepsilon$ is the category with null morphisms where:

- Objects are real vector spaces V with a seminorm $\|\cdot\|$.
- Morphisms are linear maps of norm ≤ 1 modulo maps of norm 0,

$$\mathbf{Norm}(V,W) := {\{f : V \to W \mid \|f\| \leq 1\}}\big/{\{f : V \to W \mid \|f\| = 0\}}.$$

- The null ideal is

$$\mathcal{N}_\varepsilon := \{f : V \to W \mid \|f\| \leq \varepsilon\}.$$

Idea: Kazhdan's $\|d^2\| \leq \varepsilon$ makes d^2 a null morphism.

ε-kernels and ε-cokernels

Proposition

For a morphism $f : X \to Y$ in $\mathbf{Norm}_\varepsilon$, the kernel is given by X with

$$\|x\|_{\ker(f)_\varepsilon} := \max\big(\|x\|, \varepsilon^{-1}\|f(x)\|\big).$$

Geometrically: the new unit ball is

$$X_1 \cap \varepsilon f^{-1}(Y_1).$$

ε-kernels and ε-cokernels

Proposition

For a morphism $f : X \to Y$ in $\mathbf{Norm}_\varepsilon$, the cokernel is given by Y with

$$\|y\|_{\mathrm{coker}(f)_\varepsilon} := \inf_{x \in X} (\|y - f(x)\| + \varepsilon \|x\|).$$

Geometrically: the new unit ball is

$$\mathrm{conv}(\varepsilon^{-1} f(X_1) \cup Y_1),$$

possibly plus boundary points.

Exactness

Definition

A composable pair

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is **exact** if gf is null and $\ker(g) = \ker(\mathrm{coker}(f))$.

Proposition

In $\mathbf{Norm}_\varepsilon$, exactness is equivalent to

$$\inf_{x \in X} (\|y - f(x)\| + \varepsilon\|x\|) \leq \max(\varepsilon\|y\|, \|g(y)\|) \qquad \forall y \in Y.$$

Intuition: the left-hand side measures “how far is y from being a boundary”, the right-hand side “how far is y from being a cycle”.

Exactness

Proposition

In $\mathbf{Norm}_\varepsilon$, exactness is equivalent to

$$\inf_{x \in X} (\|y - f(x)\| + \varepsilon \|x\|) \leq \max(\varepsilon \|y\|, \|g(y)\|) \qquad \forall y \in Y.$$

Compare with Kazhdan's version:

$$\forall y \, \exists x : \quad \|y - f(x)\| \leq \varepsilon \|y\| + \|g(y)\| \quad \& \quad \|x\| \leq \|y\|.$$

$\Rightarrow$ Our version may be similar enough to serve the same purpose.

However

Proposition

$\mathbf{Norm}_\varepsilon$ is not a homological category for any $\varepsilon \in (0, 1)$.

The reason is that $\mathbf{Norm}_\varepsilon$ fails the following axiom:

- Every null morphism factors through a null identity morphism.

Arrow categories to the rescue?

- Pairs (of spaces) play an important role in algebraic topology.
- Working with pairs, or more generally arrows, has better homological properties!

Theorem

Let C be any category with null morphisms having kernels and cokernels. Then **the arrow category** $\mathsf{C}^{\rightarrow}$ **is a homological category**.

Long exact sequences in the arrow category

Consider now a short exact sequence of chain complexes in C,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle i_{n+1}} & & \big\downarrow{\scriptstyle i_n} & & \big\downarrow{\scriptstyle i_{n-1}} & & \\
\cdots & \longrightarrow & Y_{n+1} & \xrightarrow{d_{n+1}} & Y_n & \xrightarrow{d_n} & Y_{n-1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle q_{n+1}} & & \big\downarrow{\scriptstyle q_n} & & \big\downarrow{\scriptstyle q_{n-1}} & & \\
\cdots & \longrightarrow & Z_{n+1} & \longrightarrow & Z_n & \longrightarrow & Z_{n-1} & \longrightarrow & \cdots
\end{array}$$

Only the differentials on $Y_\bullet$ are named, as the others are induced.

Long exact sequences in the arrow category

This induces a sequence of homology objects in $\mathsf{C}^{\to}$,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \mathrm{Ker}(q_n d_{n+1}) & \xrightarrow{\partial_{n+1,0}} & \mathrm{Ker}(d_n i_n) & \longrightarrow & \mathrm{Ker}(d_n) & \longrightarrow & \mathrm{Ker}(q_{n-1} d_n) & \xrightarrow{\partial_{n,0}} & \mathrm{Ker}(d_{n-1} i_{n-1}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathrm{Coker}(q_{n+1} d_{n+2}) & \xrightarrow{\partial_{n+1,1}} & \mathrm{Coker}(d_{n+1} i_{n+1}) & \longrightarrow & \mathrm{Coker}(d_{n+1}) & \longrightarrow & \mathrm{Coker}(q_n d_{n+1}) & \xrightarrow{\partial_{n,1}} & \mathrm{Coker}(d_n i_n) & \longrightarrow \cdots
\end{array}$$

where every “double diagonal” is null.

The sequence is exact under certain additional modularity conditions.

Long exact sequences in the arrow category

In the abelian case, we recover the usual long exact sequence by image factorization of the vertical arrows:

$$
\begin{array}{ccccccccccccc}
\dots & \longrightarrow & \mathrm{Ker}(q_n d_{n+1}) & \xrightarrow{\partial_{n+1,0}} & \mathrm{Ker}(d_n i_n) & \longrightarrow & \mathrm{Ker}(d_n) & \longrightarrow & \mathrm{Ker}(q_{n-1} d_n) & \xrightarrow{\partial_{n,0}} & \mathrm{Ker}(d_{n-1} i_{n-1}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_{n+1}(Z) & \xrightarrow{\partial_{n+1}} & H_n(X) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Z) & \xrightarrow{\partial_n} & H_{n-1}(X) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{Coker}(q_{n+1} d_{n+2}) & \xrightarrow{\partial_{n+1,1}} & \mathrm{Coker}(d_{n+1} i_{n+1}) & \longrightarrow & \mathrm{Coker}(d_{n+1}) & \longrightarrow & \mathrm{Coker}(q_n d_{n+1}) & \xrightarrow{\partial_{n,1}} & \mathrm{Coker}(d_n i_n) & \longrightarrow & \dots
\end{array}
$$

A nonabelian example

Let $K \triangleright H \triangleright G$ be normal subgroups such that K is not normal in G.

Consider the short exact sequence of chain complexes:

$$
\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & K & \hookrightarrow & G & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & K & \xrightarrow{0} & G/H & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}
$$

A nonabelian example

If the standard long exact sequence held, it would be

$$\cdots \longrightarrow K \xhookrightarrow{\partial} H \longrightarrow G/\langle K\rangle \longrightarrow G/H \xrightarrow{\partial} 0 \longrightarrow \cdots$$

where $\langle K\rangle \triangleright G$ is the normal subgroup generated by K.

But this is not exact at H!

Our long exact sequence in the arrow category:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & K & \xhookrightarrow{\partial_0} & H & \hookrightarrow & G & = & G & \xrightarrow{\partial_0} & 0 & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K & \xhookrightarrow{\partial_1} & G & \twoheadrightarrow & G/\langle K\rangle & \twoheadrightarrow & G/H & \xrightarrow{\partial_1} & 0 & \longrightarrow & \cdots
\end{array}$$

Summary: homological algebra up to ε

- Kazhdan's cohomologically flavoured techniques on ε-representations hint at a mysterious "homological algebra up to ε".
- My proposal of using $\mathbf{Norm}_\varepsilon$ can be thought of as **quantitative homological algebra** with some inherent fuzziness.
- It is plausible that reasoning in $\mathbf{Norm}_\varepsilon$ can reproduce Kazhdan's method as part of a general framework.

Summary: homological algebra with arrow categories

- However, $\mathbf{Norm}_\varepsilon$ may lack certain properties that make homological algebra well-behaved in general.
- In such situations, one can still move to the arrow category!
- This strategy may be interesting in general.
- Instead of a single homology object, one obtains the arrow

$$\text{cycles} \longrightarrow {}^{\text{chains}}\!/_{\text{boundaries}} \,.$$

- Working example: homological algebra with nonabelian groups.