What are quantum measurable spaces?

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Why measurable spaces?

Definition

Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces. Then a **Markov kernel** is

$$\begin{aligned} f \;:\; \Sigma_Y \times X &\longrightarrow [0,1] \\ (S, x) &\longmapsto f(S|x) \end{aligned}$$

such that:

- $f(\_|x)$ is a probability measure on (Y, Σ_Y) for every $x \in X$.
- $f(S|\_)$ is measurable for every $S \in \Sigma_Y$.

- Many other names: stochastic map, statistical model, communication channel, information transformer, ...
- Often no single dominating measure. (Identity kernel, sampling kernel, ...)

Measurements

Two meaningful ways to define measurements in quantum theory:

- As POVMs, i.e. algebra-valued measures $\Sigma_X \to A$,
- As channels to classical systems.

These two *should* be equivalent!

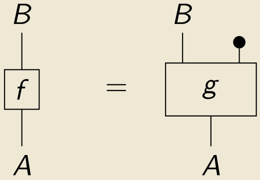
Goal

Find notions of "channel" and "system" that make this work.

Tensor dilations, graphically

Definition

A **tensor dilation** of a ucp map $f : B \to A$ is a ucp map $g : B \otimes E \to A$ such that



$$f(b) \quad = \quad \psi(b \otimes 1) \quad \forall b \in B$$

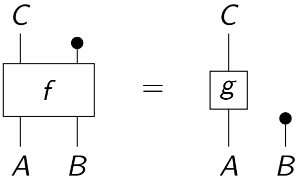
Theorem

(a) Every ucp map between matrix algebras has an initial tensor dilation.

(b) Every Markov kernel between standard Borel spaces has an initial tensor dilation.

Tensor dilations, graphically

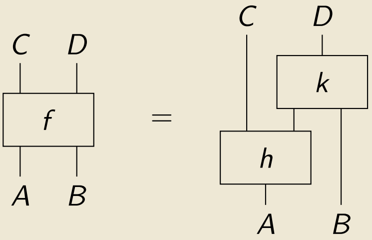
$f : C \otimes D \to A \otimes B$ is **no-signaling** if there is $g : C \to A$ such that



$$f(c \otimes 1) = g(c) \otimes 1$$

Theorem (Houghton-Larsen)

If "nice" initial dilations exist, then every no-signalling map factors as



σ-normality

Definition (e.g. Wright)

A ucp map $\varphi\colon A \to B$ is σ-**normal** if it preserves suprema of increasing sequences in A_{sa}.

- ▷ Terminology: σ-ucp, σ-homomorphism, σ-state.

Proposition

For measurable spaces (X, Σ_X) and (Y, Σ_Y), there is a bijection between

(a) Markov kernels $(X, \Sigma_X) \to (Y, \Sigma_Y)$, and

(b) σ-ucp maps $\mathcal{L}^\infty(Y, \Sigma_Y) \to \mathcal{L}^\infty(X, \Sigma_X)$.

- ▷ **No quotient by almost everywhere equality!**

σ-normality

Lemma

For every C*-algebra A, the σ-states form a face

$$\mathcal{S}_\sigma(A) \subseteq \mathcal{S}(A).$$

- Hence a σ-state is pure iff it is pure as a state.

Theorem

For every C*-algebra A, the canonical map

$$A \longrightarrow C(\mathcal{S}(A))$$

is σ-ucp.

Towards quantum measurable spaces

Goal

Find a definition of the form

$$\text{quantum measurable space} := \text{C*-algebra with extra property}$$

such that the commutative quantum measurable spaces are the C*-algebras isomorphic to $\mathcal{L}^\infty(X, \Sigma_X)$ for (suitably nice) measurable spaces (X, Σ_X).

σC*-algebras

Definition (Wright)

A σ**C*-algebra** is a C*-algebra A in which every bounded monotone sequence in A_{sa} has a supremum.

For example, every $\mathcal{L}^\infty(X, \Sigma_X)$ is a σC*-algebra.

Nice properties similar to von Neumann algebras:

- Compressions $x \mapsto axa^*$ are σ-normal.
- Borel functional calculus.

Theorem

The category of commutative σC*-algebras is equivalent to the category of Boolean σ-algebras.

General σC*-algebras are too broad

Example (cf. Sikorski)

Let $\mathcal{M} \subseteq \mathcal{L}^\infty([0,1])$ be the ideal of meagerly supported functions. Then $\mathcal{L}^\infty([0,1])/\mathcal{M}$ has no σ-states.

We want separation by (pure?) σ-states: for every $a \in A_+$,

$$\|a\| \overset{!}{=} \sup_{\phi \in \mathcal{S}_\sigma(A)} \phi(a).$$

Pedersen–Baire envelopes

Definition (Saito, Wright)

For a C*-algebra A, its **Pedersen–Baire envelope** A^∞ is the σ-closure of A inside A^{**}.

- For compact Hausdorff X,

$$C(X)^\infty \cong \mathcal{L}^\infty(X, \mathrm{Baire}(X)).$$

- For separable $\mathcal{H}$,

$$(\mathcal{K}(\mathcal{H}) + \mathbb{C})^\infty \cong \mathcal{B}(\mathcal{H}).$$

- In general, A^∞ is the universal σC*-algebra generated by A.

Pedersen–Baire envelopes

PB envelopes are good candidates for quantum measurable spaces:

- Commutative case makes sense.
- Separation by σ-states holds.
- Drawback: being a PB envelope looks like extra structure, not a property.

Lifting properties

A good definition should also be informed by what is needed for (quantum) probability theory.

Disintegration of measures is based on lifting properties:

Theorem (von Neumann, Ionescu Tulcea)

Let (X, Σ_X, μ_X) be a complete probability space. Then the canonical quotient map

$$\mathcal{L}^\infty(X, \Sigma_X) \longrightarrow L^\infty(X, \Sigma_X, \mu_X)$$

has a $*$-homomorphic section.

Can we have something similar in the quantum case?

Tensor products

- We would like to have a tensor product $\hat{\otimes}$ with

$$\mathcal{L}^\infty(X, \Sigma_X) \,\hat{\otimes}\, \mathcal{L}^\infty(Y, \Sigma_Y) \;\cong\; \mathcal{L}^\infty(X \times Y, \Sigma_X \otimes \Sigma_Y).$$

- The usual C*-algebra tensor products are too small!

- For example, the indicator function of the diagonal witnesses

$$\mathcal{L}^\infty(\mathbb{N}) \otimes_{\mathsf{min}} \mathcal{L}^\infty(\mathbb{N}) \;\subsetneq\; \mathcal{L}^\infty(\mathbb{N} \times \mathbb{N}).$$

Summary

- Surprisingly, we don't yet seem to have a good notion of quantum measurable space.
- Having one is important for foundational developments of quantum probability.
- σC*-algebras are a starting point, but too broad.