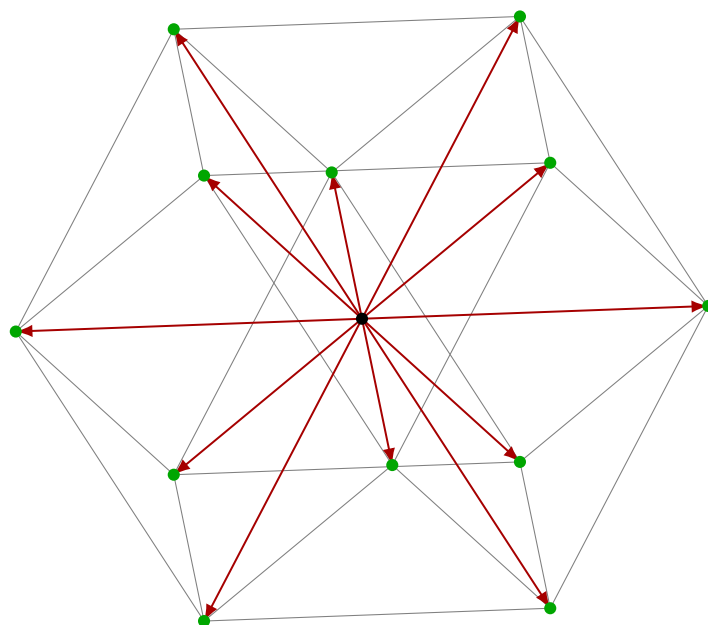


Lie Groups and Lie Algebras

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The root system of $\mathfrak{su}(3)$.

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Preface

Welcome to the course *Lie Groups and Lie Algebras*! We will delve into a rich and beautiful area of mathematics distinguished by the following features:

- ▷ A synergistic interplay between algebraic structures, analytical methods and geometric intuition.
- ▷ Strong and nontrivial classification results, such as the classification of simple Lie algebras into four infinite families and five exceptional cases.
- ▷ A wide range of applications in mathematics and physics. Especially in particle physics, Lie theory seems to be a fundamental ingredient of the laws of nature!

Given the breadth and depth of the subject, we will not be able to cover everything in detail. The goal of the course is to provide a broad overview of the most important concepts and results, with focus on developing intuition and getting some understanding of the applications to physics.

Prerequisites

Beyond some degree of mathematical maturity, we assume familiarity with:

- ▷ Linear algebra, including inner products, orthonormal bases and the spectral theorem.
- ▷ Basic concepts and results of analysis, such as criteria for the convergence of series.
- ▷ Basic concepts of group theory.

Some background in differential geometry or physics may be helpful, but is not required.

Course structure

Each week is divided into a **problem session** and a **lecture**:

- ▷ The problem session is intended to give you the opportunity to develop as much of the material as possible on your own. Independent discovery is both the most effective and most fun way to learn a subject. To the extent that this is feasible, the problems will be designed as guided research problems rather than conventional homework exercises. Solutions will be provided after each problem session, and your own solutions are **not graded or evaluated** in any way. You are encouraged to work together and share your understanding and ideas. In the spirit of doing your own discovery, the idea is to approach the problems **without internet access or other external resources**.

There is no homework in the usual sense. However, it is strongly recommended that you:

- (a) **Before each problem session**, make sure to review the “particular prerequisites” listed at the beginning of the chapter.
- (b) **After each problem session**, make sure that you understand the solutions and study the lecture notes. Time permitting, you may want to do this *before* the lecture.

- ▷ The lecture following each problem session consolidates and builds on the material developed in the problem session.

Our focus is going to be on conceptual understanding, the development of intuition and working with examples, rather than on proofs and technicalities. In this spirit, we will cover proofs only when they provide relevant insight.

Literature

- ▷ Brian C. Hall's *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* [1].

Potential further of topics

Further possibilities beyond the current table of contents (for my own reference) include:

- ▷ Categories of representations, tensor product of representations and string diagrams
- ▷ Dyson/Magnus series
- ▷ Lorentz group and its representation theory
- ▷ Universal cover and projective representations, belt trick
- ▷ Universal enveloping algebra and Casimir elements
- ▷ Clifford algebras
- ▷ Exceptional Lie algebras / Lie groups
- ▷ Complexification and real forms
- ▷ Nilpotence, solvability, semisimplicity
- ▷ Space groups

Chapter 1

The rotation group

Particular prerequisites:

- ▷ Linear maps and their matrix representation.
- ▷ Inner products, Euclidean norm, orthonormal bases.
- ▷ The definition of group.

1.1 Problems

We let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^3$, that is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Recall that every linear map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by a matrix $A \in \mathbb{R}^{3 \times 3}$. We say that such a map *preserves lengths* if

$$\|Ax\| = \|x\| \quad \forall x \in \mathbb{R}^3.$$

We write

$$O(3) := \{A \in \mathbb{R}^{3 \times 3} \mid A \text{ preserves lengths}\}$$

where the letter “ O ” hints at the words “orthogonal” or “orthonormal”, which will become clear in Problem 1.1.3.

Problem 1.1.1. (a) Show the set of all length-preserving maps is a group under matrix multiplication: if A and B are length-preserving, then so are AB and A^{-1} .

(b) Would we also get a group if we considered length-preserving maps for a different norm?

As is often the case, it is helpful to look at some concrete examples before analyzing the general structure further. For example, the following maps preserve lengths:

- ▷ Reflection at the origin.
- ▷ Rotation around the x_1 -axis with angle $\varphi \in (0, 2\pi)$.
- ▷ Reflection at the x_2x_3 -plane.
- ▷ Reflection at the x_1 -axis.

Problem 1.1.2. (a) What are the matrix representations of these maps?

(b) Are they all distinct maps? Or is there a reflection that can also be seen as a rotation?

With these examples at hand, we can aim for some understanding of length-preserving maps in general.

Problem 1.1.3. Show that the following are equivalent for every matrix $A \in \mathbb{R}^{3 \times 3}$:

(a) A preserves lengths.

(b) A preserves the inner product:

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^3.$$

(c) The columns of A form an orthonormal basis of $\mathbb{R}^3$.

(d) A maps every orthonormal basis of $\mathbb{R}^3$ to an orthonormal basis.

(e) $A^t A = I$.

Soon we will obtain some understanding of what general elements of $O(3)$ look like. But this requires some preparation.

1.1.4 Definition. For a group G , elements $A, B \in G$ are **conjugate** if there is $C \in G$ such that

$$B = C^{-1}AC.$$

Problem 1.1.5. (a) Show that conjugacy is an equivalence relation.

(b) Convince yourself that $A, B \in O(3)$ are conjugate if and only if B is the matrix representation of A in another orthonormal basis.

1.1.6 Definition. The equivalence classes of conjugacy are called **conjugacy classes**.

The conjugacy classes in $O(3)$ therefore classify length-preserving maps up to basis. Describing these conjugacy classes explicitly is an important problem that we will address in the lecture.

Here is one more concept that will play a central role throughout Lie theory.

1.1.7 Definition. For a group G and $A, B \in G$, the **commutator** is

$$A^{-1}B^{-1}AB.$$

The following question provides a glimpse of why commutators are important.

Problem 1.1.8. Visualize the commutator of the following two elements of $O(3)$ informally:

▷ A = a small rotation around the x_1 -axis,

▷ B = a small rotation around the x_2 -axis.

Hint: No calculation is required; you can do this with your hands or some other item that you rotate. Follow the trajectory of a point on the item as you apply the four rotations in the order given by the commutator. It's not easy!

The problem and its solution are currently informal; over the course of the next few weeks, we will develop the tools to make understand this more rigorously.

1.2 Lecture

Note: The reasoning in this lecture is going to be especially informal and not intended as mathematically precise at all. Although all mathematical statements are correct, the justifications that we give are often not formal proofs. Developing everything in full rigor would take us too far afield and would not be worth the effort.

The **orthogonal group**

$$O(3) = \{A \in \mathbb{R}^{3 \times 3} \mid A^t A = I\}$$

can be thought of as the **symmetry group of physical space**. In making this statement, we view physical space as a three-dimensional vector space equipped with the Euclidean notion of length. In particular, every rotation of a physical object around its center of mass is an element of $O(3)$.

Composing two or more such rotations gives us new rotations. This is a very important fact in some situations! For example, an aircraft has three principal axes of rotation: roll, pitch and yaw (Fig. 1.1). The fact that that every rotation can be achieved by composing these is the very reason why an aircraft is built exactly such that these three rotations can be controlled independently. A pilot must have a good intuition for how to do this.

Let us now formalize this a bit more. To start with, how do we actually define what a rotation is? Our intuitive understanding suggests that there should be an **axis of rotation**, which we take to be a unit vector $u \in \mathbb{R}^3$. Suppose that we now complete u to an orthonormal basis (u, v, w) of $\mathbb{R}^3$. Then in such a basis, a rotation around the axis u which rotates v in the direction towards w by an angle φ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}. \quad (1.2.1)$$

So let us simply use this as the definition.

1.2.1 Definition. A **rotation** is a linear map $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which in some orthonormal basis has a matrix representation of the form (1.2.1).

Clearly the columns of (1.2.1) are orthonormal, and therefore this matrix belongs to $O(3)$. If we write C for the matrix whose columns are the vectors of the orthonormal basis (u, v, w) , then this is the change of basis matrix from the standard basis to (u, v, w) . The definition thus says that we should have

$$A = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix} C^{-1}$$

for some $C \in O(3)$. But since all terms on the right-hand side are elements of $O(3)$ and $O(3)$ is a group, we can conclude that every rotation A is itself an element of $O(3)$. Informally, this is obvious: every rotation preserves lengths, and length preservation is how we had defined $O(3)$.

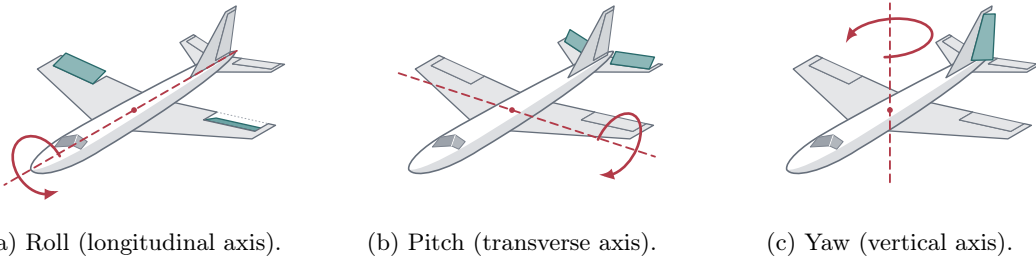
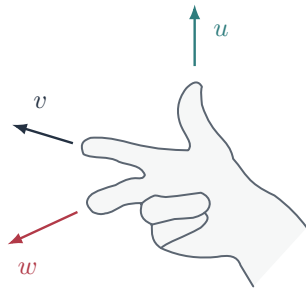
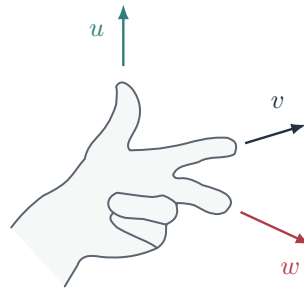


Figure 1.1: The three principal rotations of an aircraft. The dashed axes pass through its center of mass; the corresponding control surfaces are indicated, with angles exaggerated for visibility.



(a) Positive orientation (right hand).



(b) Negative orientation (left hand).

Figure 1.2: The two possible orientations of an orthonormal basis (u, v, w) . The thumb, index finger and middle finger indicate the directions of u , v and w , respectively.

1.2.2 Remark. A rotation with an angle $\varphi = \pi$ can equivalently be seen as a *reflection* at its axis, as we already noted in the solution to Problem 1.1.2(b).

So how unique are the axis u and the angle φ of a rotation? To address this question, it is useful to first consider how unique the choice of the orthonormal basis (u, v, w) is. For a given u , the second vector v can be any unit vector perpendicular to u . Its choice does not matter in our context: the rotation should rotate *all* vectors perpendicular to u by the same angle, namely the angle φ in the positive direction as seen from the tip of u looking towards the origin.¹

Once we have fixed u and v , then what about the choice of w ? Now we have *two* possibilities for w that differ by a sign. The two result bases differ by their **orientation** in the sense of Fig. 1.2. If we then consider the rotation (1.2.1) in the two bases, it is intuitively clear that these two rotations are the same, except that they rotate in opposite directions, which has the same effect as replacing φ by $-\varphi$, or equivalently by $2\pi - \varphi$. Therefore we can restrict to positively oriented bases without loss of generality,

Moreover, we can restrict to angles $\varphi \in [0, \pi]$ without loss of generality, because a rotation by an angle $\varphi \in (\pi, 2\pi)$ is the same as a rotation by $-\varphi$, or equivalently by $2\pi - \varphi \in (0, \pi)$, with respect to the opposite axis $-u$.

We then have the following characterization of rotations in $O(3)$, which we will use without proof.²

1.2.3 Theorem. (a) Together with the identity map, the rotations form a subgroup of $O(3)$ given by the **rotation group** or **special orthogonal group**

$$SO(3) = \{A \in O(3) \mid \det A = 1\}. \quad (1.2.2)$$

(b) For a given rotation A , there is a unique angle of rotation $\varphi \in (0, \pi]$. With such φ , the axis of rotation u is

- ▷ strictly unique as soon as $\varphi < \pi$, and
- ▷ unique up to sign if $\varphi = \pi$.

1.2.4 Remark. A priori, it is not even obvious that the composition of two rotations is again a rotation (or the identity). But this is guaranteed by (a): the multiplicativity of the determinant

$$\det(AB) = \det(A)\det(B)$$

already implies that the subset of $O(3)$ with determinant 1 is closed under multiplication and inverses and hence a subgroup. Also it is easy to see that every rotation (1.2.1) has determinant 1. What is more difficult is to show that every element of $O(3)$ with determinant 1 is a rotation, and this is the deeper fact that needs linear algebra theorems.

¹This is of course entirely conventional, but it is useful to agree on a convention.

²Proofs of statements like this can be found in many linear algebra textbooks. One elegant approach is via the spectral theorem applied to the matrices A seen as matrices with *complex* entries; see [3, Section 6.5, proof of Theorem 6.5.1].

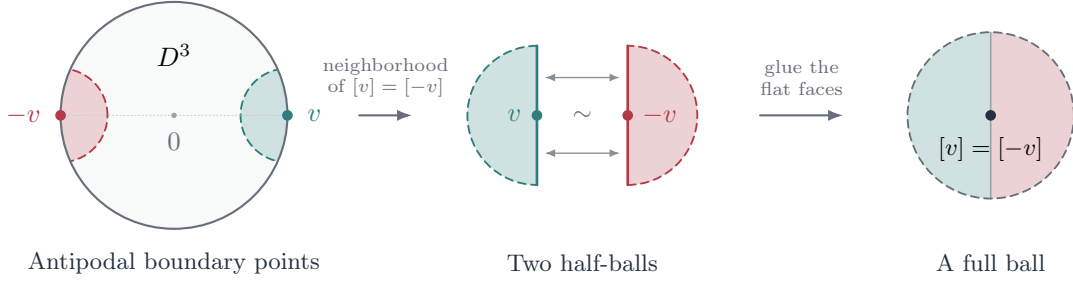


Figure 1.3: A neighborhood of a boundary point $[v] = [-v]$ in $D^3/\sim$, depicted in a two-dimensional slice. The two boundary neighborhoods become topological half-balls. Identifying their flat faces produces a full three-dimensional ball around $[v] = [-v]$.

Topology of the rotation group. By virtue of being a subset of $\mathbb{R}^{3 \times 3} \cong \mathbb{R}^9$, the rotation group $SO(3)$ inherits a topology from $\mathbb{R}^9$. Can we describe this topology more explicitly?

1.2.5 Proposition. $SO(3)$ is homeomorphic to $D^3/\sim$, where

- ▷ $D^3 = \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$ is the closed unit ball, and
- ▷ $\sim$ is the equivalence relation that identifies antipodal points on the boundary unit sphere S^2 .

Sketch of proof. We define a map

$$SO(3) \longrightarrow D^3/\sim$$

which assigns to a rotation with axis $v \in S^2$ and angle $\varphi \in (0, \pi]$ the vector

$$\frac{\varphi v}{\pi} \in D^3.$$

Thus rotations by a small angle are closer to the origin, and the origin itself corresponds to the identity rotation. It is informally clear that this map is continuous, since varying the axis and angle of rotation continuously also varies the matrix continuously. Since it is bijective by the uniqueness of axis and angle as in Theorem 1.2.3(a), we are dealing with a continuous bijective map.

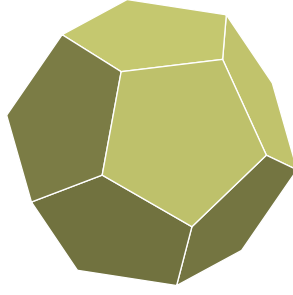
Although the inverse map of a continuous bijection is not always continuous, in this case both spaces involved are compact and Hausdorff. In such a situation, the inverse is automatically continuous [2, Theorem 26.6], which means that we have a homeomorphism. $\square$

In fact, $SO(3)$ is actually a **manifold**: every point has a neighborhood that looks like the open unit ball in $\mathbb{R}^3$. This is obvious for points in the interior of D^3 , corresponding to rotations by angles $\varphi < \pi$. But it also holds for the reflections on the boundary S^2 : there, every point has a neighborhood that looks like a half-ball on one side of D^3 and a half-ball on the other side, which together form a full ball (Fig. 1.3). The fact that this manifold is *three-dimensional* is essentially why our aircraft has *three* generating axes of rotation.

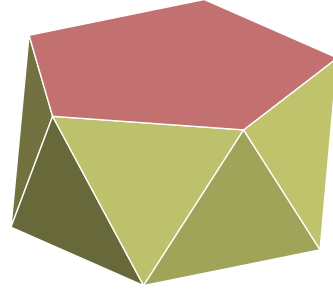
Also worth noting is that $SO(3)$ is *connected*: there is a continuous path between any two points. Since a physical object like an aircraft is limited to continuous changes of orientation, this connectedness is a prerequisite for the object to be able to reach any orientation from any other orientation.

1.2.6 Remark. As we already know, the matrices in $O(3)$ are those matrices that have orthonormal columns. One can show that $SO(3)$ corresponds precisely to those matrices in $O(3)$ that have orthonormal and *positively oriented* columns.

Hence the same statements apply to the topological space of positively oriented orthonormal bases of $\mathbb{R}^3$: it is a three-dimensional connected manifold, and it is homeomorphic to $D^3/\sim$.



(a) Dodecahedron.



(b) Pentagonal antiprism.³

Figure 1.4: Two polyhedra.

Rotation-reflections. If we look at the three reflections in Problem 1.1.2(b), we can note that only the reflection at an axis belongs to $SO(3)$. The other two have determinant -1 and are therefore not rotations.

More generally, can we describe all elements of $O(3)$ that are not in $SO(3)$? Indeed this is quite simple now based on the following observation.

1.2.7 Lemma. *For every $A \in O(3)$, we have $\det A = \pm 1$.*

Proof. From $\det(A^t A) = \det(I) = 1$ and $\det(A^t) = \det(A)$, we get $\det(A)^2 = 1$. $\square$

In particular, we must have $\det A = -1$ for every $A \in O(3) \setminus SO(3)$. If we now fix any element of $O(3) \setminus SO(3)$, then multiplication in $O(3)$ by this element implements a bijection

$$O(3) \setminus SO(3) \xrightarrow{\cong} SO(3).$$

A particularly convenient choice is to use the reflection at the origin $-I \in O(3) \setminus SO(3)$, in which case the bijection simply maps $A \mapsto -A$. In group-theoretic terms, what we are doing is to write $O(3)$ as its decomposition into **cosets** with respect to the subgroup $SO(3)$,

$$O(3) = SO(3) \sqcup -SO(3),$$

where “ $\sqcup$ ” denotes disjoint union.

Based on this idea, we can easily extend our results about $SO(3)$ to all of $O(3)$ as follows.

1.2.8 Corollary. *Every element of $O(3)$ is either a rotation or a **rotation-reflection**, by which we mean a linear map $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which in some orthonormal basis has a matrix representation of the form*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}. \quad (1.2.3)$$

Concerning the topology, $O(3)$ is homeomorphic to $(D^3/\sim) \sqcup (D^3/\sim)$.

In geometric terms, such A indeed rotates around the axis and reflects at the plane perpendicular to it; since these two operations commute, either of them can be performed first.

Proof. This follows directly from Theorem 1.2.3 and the bijection $A \mapsto -A$ between $O(3) \setminus SO(3)$ and $SO(3)$. The only subtlety is that this gives rotation-reflections in the form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & -\cos \varphi \end{pmatrix},$$

but these coincide with the above form for $\varphi \in (0, \pi]$ after replacing φ by $\pi - \varphi$. $\square$

Finally, it is worth talking about the *symmetry group* of a physical object. While we have an intuitive understanding of what the symmetries of e.g. a dodecahedron are, formalizing this intuition may not be obvious and also reveals an important subtlety.

Let us model a “physical object” as a set $P \subset \mathbb{R}^3$ of points in three-dimensional space with $0 \in P$, where we informally assume that 0 is the center of mass of the object.

1.2.9 Definition. A *symmetry* of P is $A \in O(3)$ such that $AP = P$.

In other words, a symmetry is rotation or rotation-reflection that maps the object to itself. By invertibility of such A , we automatically also have $A^{-1}P = P$. Since $AP = P$ and $BP = P$ implies

$$(AB)P = A(BP) = AP = P,$$

we can conclude that the symmetries of P form a subgroup of $O(3)$, called the **symmetry group** of P .

1.2.10 Remark. The subtlety hinted at above is the question whether we require $A \in O(3)$ or $A \in SO(3)$ in the definition of a symmetry. In other words, does a reflection count as a symmetry?

For example, one can show that the dodecahedron in Fig. 1.4a has a symmetry group consisting of 60 rotations (including the identity) and 60 rotation-reflections, such as a reflection at an equatorial plane that is parallel and in between two opposite faces. In your intuition, would you count these as symmetries of the dodecahedron, even though they cannot be realized by a physical rotation of it? Since this is not a mathematical question, there is no right or wrong answer.

The pentagonal antiprism in Fig. 1.4b illustrates the case where the same reflection does not map the object to itself, but it *does* if we also combine it with a rotation of one tenth of a turn around the vertical axis.

³Adapted from *Rotoreflexion example antiprism* by Tomruen, via Wikimedia Commons.

Chapter 2

Exponentiating matrices

Particular prerequisites:

- ▷ The exponential function, its power series and exponential law $e^{a+b} = e^a e^b$.
- ▷ The concept of absolute convergence of a series $\sum_{j=1}^{\infty} a_j$ in $\mathbb{R}^n$.
- ▷ How to prove such absolute convergence via the ratio test.

2.1 Problems

To be covered: properties including basic BCH

2.2 Lecture

Chapter 3

Lie groups

To be covered: other examples like Euclidean group, symplectic group, Heisenberg group, Lorentz group, $SL(2, \mathbb{C})$, exceptional isos?

3.1 Problems

3.2 Lecture

Chapter 4

Derivations and vector fields

4.1 Problems

4.2 Lecture

Chapter 5

Lie algebras

To be covered: Jacobi identity and its meaning, Ado's theorem

5.1 Problems

5.2 Lecture

Chapter 6

Representation theory of the Heisenberg group

6.1 Problems

6.2 Lecture

Chapter 7

Representation theory of $SU(n)$

7.1 Problems

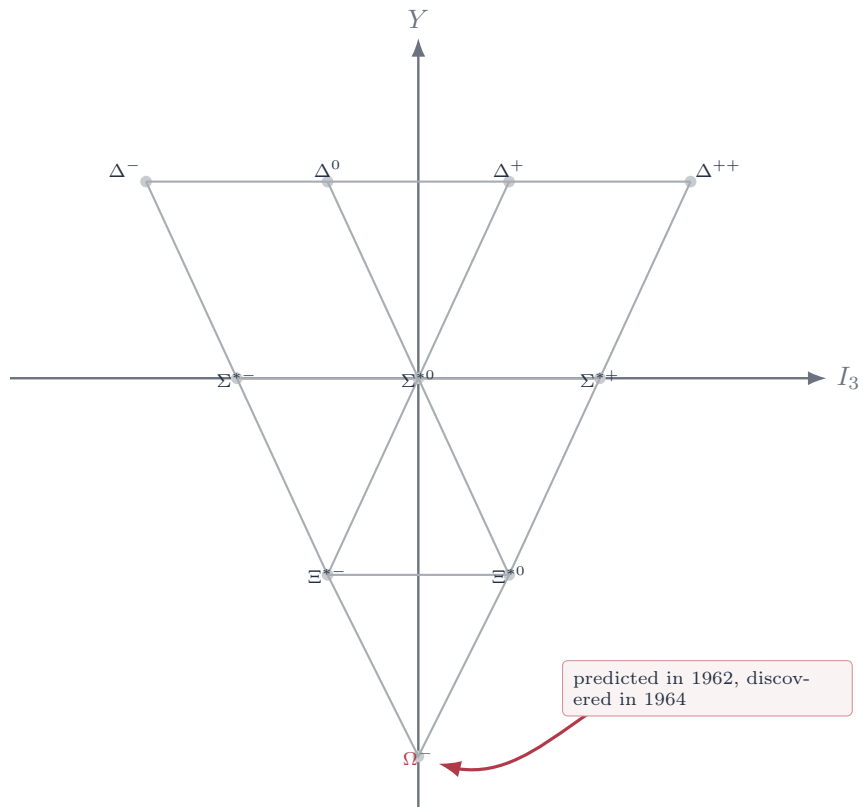


Figure 7.1: Gell-Mann's decuplet of baryon particles together with his prediction of the Ω^- particle.

7.2 Lecture

Chapter 8

Harmonic functions and the Peter–Weyl theorem

8.1 Problems

8.2 Lecture

Chapter 9

Root systems and the classification of simple Lie algebras

9.1 Problems

9.2 Lecture

Appendix A

Selected linear algebra background

Here we collect some material on linear algebra that may not be familiar to all readers.

diagonalizing commuting operators

Hilbert spaces

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